EE 213 ELECTRIC CIRCUITS II

Chapter 11
Balanced Three-Phase Circuits

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Recap: Sinusoidal Sources

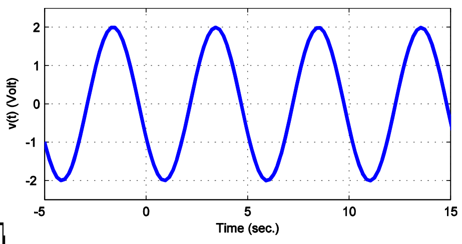
- A **sinusoidal voltage (current) source** produces a voltage (current) that varies sinusoidally in time.

$$v(t) = V_m \cos(\omega t + \phi)$$

Amplitude

Frequency (radians)

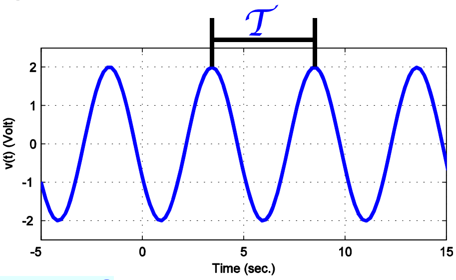
Phase



Recap: Frequency and Period

- A function that repeats itself is called **periodic.** The amount of time required to complete one full cycle is called the **period**.
- The number of cycles per second is called the **frequency**.

Period (sec.) → $\mathcal{T} = \dfrac{1}{f}$ ← Frequency (Hertz)



Frequency (rad./sec.) → $\omega = 2\pi f$

Recap: Root-Mean-Square (RMS) Value

- The rms value of a periodic function is defined as the **square-root** of the **mean** value of the **squared** function.

$$V_{rms} = \sqrt{\frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} v^2(t)dt}$$

- For sinusoidal signal sources, the RMS value can be shown to be

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

Recap: Phasor

- **Phasor** is a complex number that carries the **amplitude and phase** information of a sinusoidal function.

$$\mathcal{V} = \mathcal{P}\{\mathcal{V}_m cos(\omega t + \phi)\} = \mathcal{V}_m e^{j\phi}$$

- The sinusoidal function is obtained from the inverse phasor transform as

$$v(t) = \mathcal{P}^{-1}\{\mathcal{V}\} = \Re e\{\mathcal{V} e^{j\omega t}\}$$

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Recap: Polar and Rectangular Forms

- **Euler Identity** shows us how to go from polar form to the rectangular form:

$$e^{j\phi} = cos(\phi) + j\, sin(\phi)$$

- One can then go from rectangular to polar form as

$$\mathcal{V} = x + jy = \mathcal{V}_m e^{j\phi}$$

with

$$\mathcal{V}_m = \sqrt{x^2 + y^2} \qquad \phi = tan^{-1}\left(\frac{y}{x}\right)$$

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Recap: Addition and Multiplication

- Multiplication is more convenient in polar form:

$$\underbrace{\mathcal{V}_{m1}e^{j\phi_1}}_{\mathcal{V}_1} \cdot \underbrace{\mathcal{V}_{m2}e^{j\phi_2}}_{\mathcal{V}_2} = \underbrace{\mathcal{V}_{m1}\mathcal{V}_{m2}}_{\mathcal{V}} e^{j\overbrace{(\phi_1+\phi_2)}^{\phi}}$$

- Taking a power is simple in polar form:

$$\left(\mathcal{V}_m e^{j\phi_1}\right)^k = \mathcal{V}_m^k e^{jk\phi}$$

- Addition is more convenient in rectangular form:

$$\underbrace{\mathcal{V}_{m1}e^{j\phi_1}}_{\mathcal{V}_1} + \underbrace{\mathcal{V}_{m2}e^{j\phi_2}}_{\mathcal{V}_2} = \mathcal{V}_{m1}\cos(\phi_1) + \mathcal{V}_{m2}\cos(\phi_2) + j\left(\mathcal{V}_{m1}\sin(\phi_1) + \mathcal{V}_{m2}\sin(\phi_2)\right)$$

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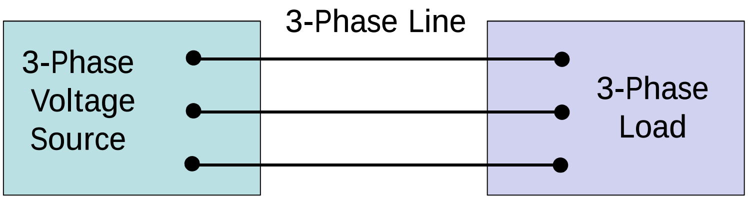
Balanced Three-Phase Voltages



- A set of balanced three-phase voltage consists of three sinusoidal voltages that have identical amplitudes and frequencies, but are out of phase with each other by exactly 120 degrees ($2\pi/3$ radians).

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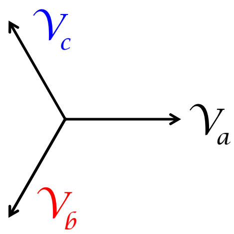
Positive and Negative Phase Sequences

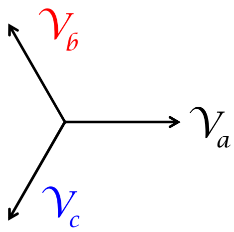
- **abc** or Positive phase sequence:

$$
\begin{aligned}
V_a &= V_m \\
V_b &= V_m e^{-j120^\circ} \\
V_c &= V_m e^{+j120^\circ}
\end{aligned}
$$



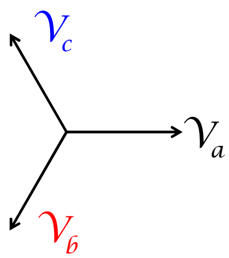
- **acb** or Negative phase sequence:

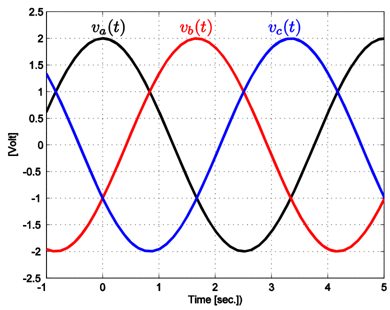
$$
\begin{aligned}
V_a &= V_m \\
V_b &= V_m e^{+j120^\circ} \\
V_c &= V_m e^{-j120^\circ}
\end{aligned}
$$



Positive Phase or abc Sequence

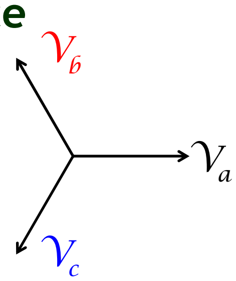
$$
\begin{aligned}
V_a &= V_m \\
V_b &= V_m e^{-j120^\circ} \\
V_c &= V_m e^{+j120^\circ}
\end{aligned}
$$

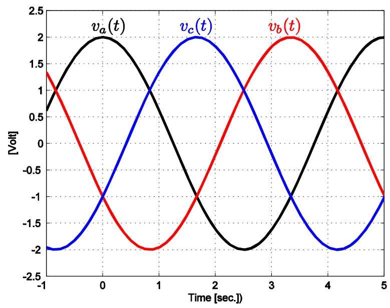




Negative Phase or acb Sequence

$$\begin{aligned} \mathcal{V}_a &= \mathcal{V}_m \\ \mathcal{V}_b &= \mathcal{V}_m e^{+j120^\circ} \\ \mathcal{V}_c &= \mathcal{V}_m e^{-j120^\circ} \end{aligned}$$





Important Properties

- The sum of balanced three-phase voltages in phasor domain is zero.

$$\mathcal{V}_a + \mathcal{V}_b + \mathcal{V}_c = 0$$

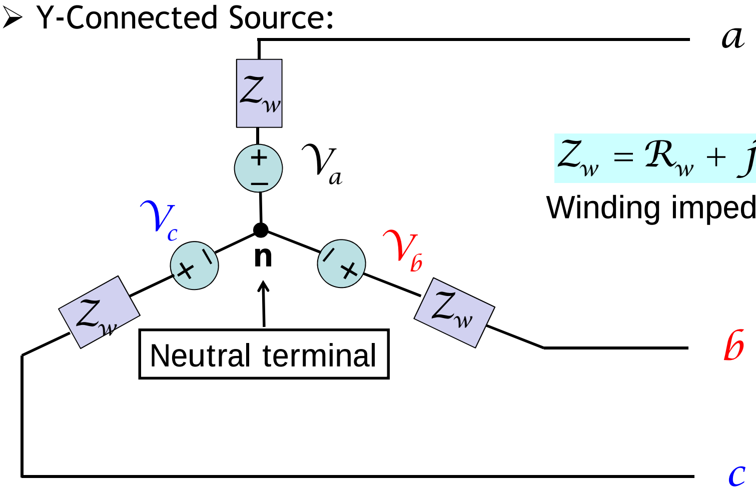
- This is true for the instantaneous voltages as well:

$$v_a(t) + v_b(t) + v_c(t) = 0$$

- If we know the phase sequence (i.e. whether it is positive or negative) and one voltage in the set, then we can determine the entire set.

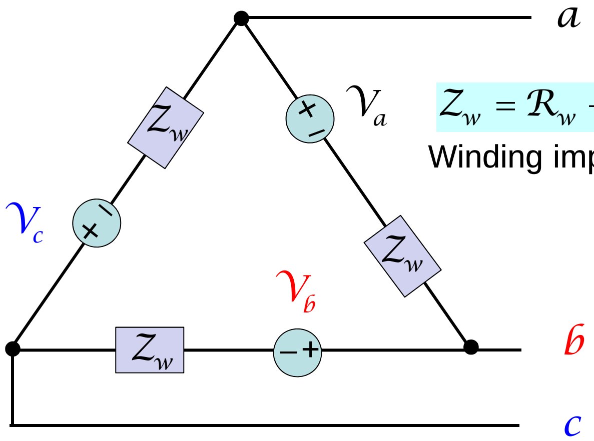
Three-Phase Voltage Sources

➢ Y-Connected Source:



Three-Phase Voltage Sources

➢ Δ-Connected Source:



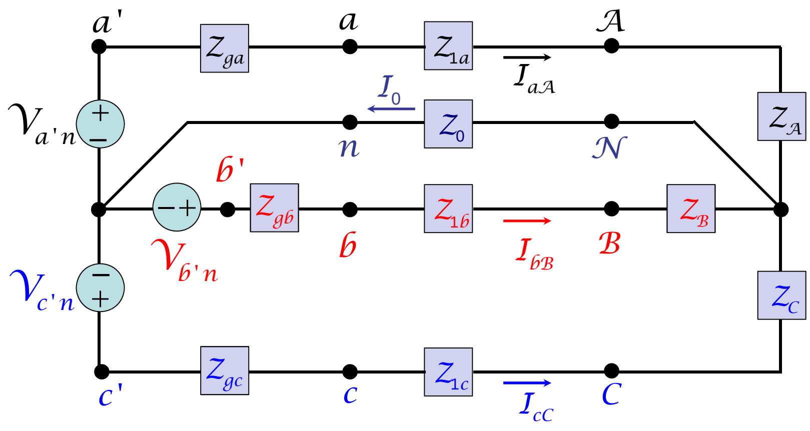
Possible Combinations of Sources & Loads

- Since both source and load can be either Y- or Δ-connected, there are four possible combinations:

Source	Load
Y	Y
Y	Δ
Δ	Y
Δ	Δ

- As other combinations can also be reduced to it, we first start with the analysis of the Y-Y combination.

Analysis of the Y-Y Circuit



$$\frac{V_{\mathcal{N}}}{Z_0} + \frac{V_{\mathcal{N}} - V_{a'n}}{Z_{\mathcal{A}} + Z_{1a} + Z_{ga}} + \frac{V_{\mathcal{N}} - V_{b'n}}{Z_{\mathcal{B}} + Z_{1b} + Z_{gb}} + \frac{V_{\mathcal{N}} - V_{c'n}}{Z_{\mathcal{C}} + Z_{1c} + Z_{gc}} = 0$$

Conditions for a Balanced 3-Phase Circuit

1. Balanced three-phase voltage sources:

$$\mathcal{V}_{a'n} + \mathcal{V}_{b'n} + \mathcal{V}_{c'n} = 0$$

2. Same impedance in each phase of the voltage source:

$$Z_{ga} = Z_{gb} = Z_{gc}$$

3. Same impedance in each line (or phase) conductor:

$$Z_{1a} = Z_{1b} = Z_{1c}$$

4. Same impedance in each phase of the load:

$$Z_{\mathcal{A}} = Z_{\mathcal{B}} = Z_{\mathcal{C}}$$

Balanced 3-Phase Y-Y Circuit

- For a balanced three-phase Y-Y circuit, we have

$$\mathcal{V}_{\mathcal{N}}\left(\frac{1}{Z_0} + \frac{3}{Z_\phi}\right) = \frac{\mathcal{V}_{a'n} + \mathcal{V}_{b'n} + \mathcal{V}_{c'n}}{Z_\phi} = 0 \quad \Rightarrow \mathcal{V}_{\mathcal{N}} = 0$$

where $Z_\phi = Z_{\mathcal{A}} + Z_{1a} + Z_{ga} = Z_{\mathcal{B}} + Z_{1b} + Z_{gb} = Z_{\mathcal{C}} + Z_{1c} + Z_{gc}$

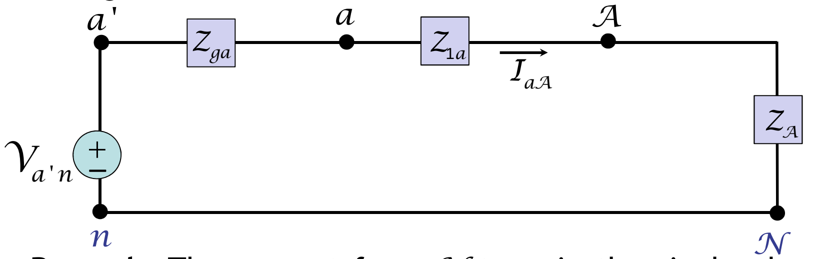
- The line currents are necessarily balanced:

$$I_{a\mathcal{A}} = \frac{\mathcal{V}_{a'n} - \mathcal{V}_{\mathcal{N}}}{Z_{\mathcal{A}} + Z_{1a} + Z_{ga}} = \frac{\mathcal{V}_{a'n}}{Z_\phi} \qquad I_{b\mathcal{B}} = \frac{\mathcal{V}_{b'n} - \mathcal{V}_{\mathcal{N}}}{Z_{\mathcal{B}} + Z_{1b} + Z_{gb}} = \frac{\mathcal{V}_{b'n}}{Z_\phi}$$

$$I_{c\mathcal{C}} = \frac{\mathcal{V}_{c'n} - \mathcal{V}_{\mathcal{N}}}{Z_{\mathcal{C}} + Z_{1c} + Z_{gc}} = \frac{\mathcal{V}_{c'n}}{Z_\phi}$$

A Single-Phase Equivalent Circuit

- $\mathcal{V}_{\mathcal{N}} = 0 \Rightarrow I_0 = 0$. Hence, we may either remove the neutral conductor or make a short circuit from n to $\mathcal{N}$.
- Analysis of a single phase is enough, since other phase voltages/currents follow from balancedness.

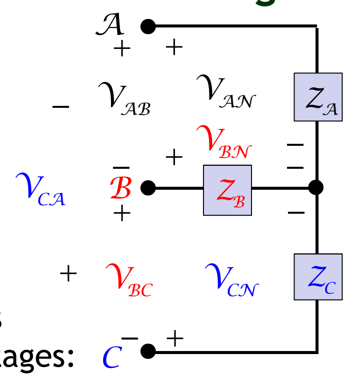


- **Remark:** The current from $\mathcal{N}$ to n in the single-phase circuit is not I_0! Recall that $I_0 = I_{a\mathcal{A}} + I_{b\mathcal{B}} + I_{c\mathcal{C}} = 0$.

Line-to-Line and Line-to-Neutral Voltages

- Line-to-line voltages:

$$\mathcal{V}_{\mathcal{AB}} = \mathcal{V}_{\mathcal{AN}} - \mathcal{V}_{\mathcal{BN}}$$
$$\mathcal{V}_{\mathcal{BC}} = \mathcal{V}_{\mathcal{BN}} - \mathcal{V}_{\mathcal{CN}}$$
$$\mathcal{V}_{\mathcal{CA}} = \mathcal{V}_{\mathcal{CN}} - \mathcal{V}_{\mathcal{AN}}$$

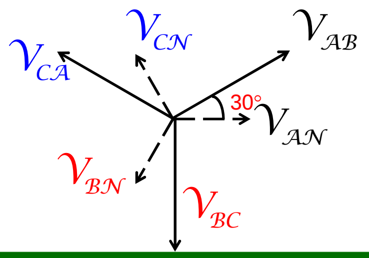


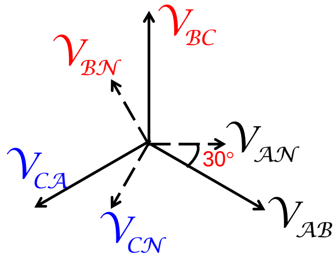
- Balanced line-to-neutral voltages lead to balanced line-to-line voltages:

$$\begin{aligned} \mathcal{V}_{\mathcal{AN}} &= \mathcal{V}_\phi \\ \mathcal{V}_{\mathcal{BN}} &= \mathcal{V}_\phi\, e^{-j120^\circ} \\ \mathcal{V}_{\mathcal{CN}} &= \mathcal{V}_\phi\, e^{+j120^\circ} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathcal{V}_{\mathcal{AB}} &= \sqrt{3}\mathcal{V}_\phi\, e^{+j30^\circ} \\ \mathcal{V}_{\mathcal{BC}} &= \sqrt{3}\mathcal{V}_\phi\, e^{-j90^\circ} \\ \mathcal{V}_{\mathcal{CA}} &= \sqrt{3}\mathcal{V}_\phi\, e^{+j150^\circ} \end{aligned}$$

Balancedness of Line-to-Line Voltages

- Line-to-line voltages form a set of balanced three-phase voltages.
- The magnitude of the line-to-line voltage is $\sqrt{3}$ times the magnitude of the line-to-neutral voltage.
- Positive- (negative-) phase sequence line-to-line voltages lead (lag) line-to-neutral voltages by 30°.

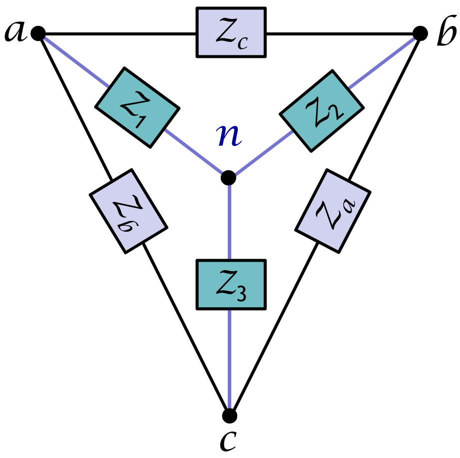




Some Terminology and Remarks

- **Line voltage:** voltage accross any pair of lines
- **Phase voltage**: voltage accross a single phase
- **Line current**: current in a single line
- **Phase current**: current in a single phase
- In a Y-connection, line current and phase current are identical.
- In a Δ-connection, line voltage and phase voltage are identical.

Recap: Δ-to-Y Transformations



Δ-to-Y:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_b}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Y-to-Δ:

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2}$$

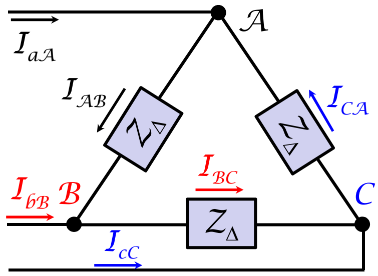
$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3}$$

Equal impedances $\Rightarrow \; Z_Y = \dfrac{Z_\Delta}{3}$

Analysis of the Y-Δ Circuit

- Y-to-Δ transformation in the load → single-phase equivalent circuit in Slide 19 with $Z_{\mathcal{A}}=Z_\Delta/3$.
- Line currents:

$$I_{a\mathcal{A}} = I_{\mathcal{AB}} - I_{\mathcal{CA}}$$
$$I_{b\mathcal{B}} = I_{\mathcal{BC}} - I_{\mathcal{AB}}$$
$$I_{c\mathcal{C}} = I_{\mathcal{CA}} - I_{\mathcal{BC}}$$

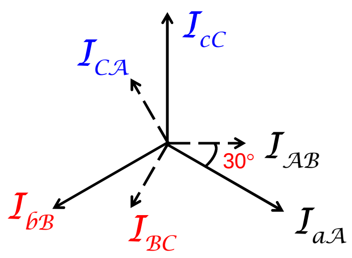


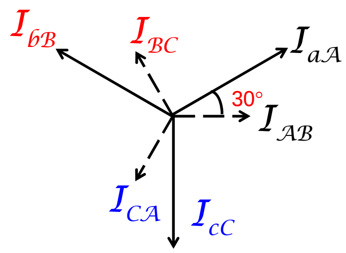
- Balanced phase currents lead to balanced line currents

$$\begin{aligned} I_{\mathcal{AB}} &= I_\phi \\ I_{\mathcal{BC}} &= I_\phi e^{-j120^\circ} \\ I_{\mathcal{CA}} &= I_\phi e^{+j120^\circ} \end{aligned} \quad \Rightarrow \quad \begin{aligned} I_{a\mathcal{A}} &= \sqrt{3} I_\phi e^{-j30^\circ} \\ I_{b\mathcal{B}} &= \sqrt{3} I_\phi e^{-j150^\circ} \\ I_{c\mathcal{C}} &= \sqrt{3} I_\phi e^{+j90^\circ} \end{aligned}$$

Balancedness of Line Currents

- Line currents form a set of balanced three-phase currents.
- The magnitude of the line currents is √3 times the magnitude of the phase currents.
- Positive- (negative-) phase sequence line currents lag (lead) the phase currents by 30°.

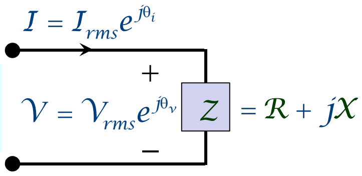




Recap: Power Calculations

- Complex power:

$$S = V I^* = |I|^2 Z = \frac{|V|^2}{Z^*} = P + jQ \quad (\text{VA})$$



- Average power: $P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) \quad (\text{W})$
  $= I_{rms}^2 R = \frac{V_{rms}^2}{R}$
- Reactive power: $Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i) \quad (\text{VAR})$
  $= I_{rms}^2 X = \frac{V_{rms}^2}{X}$
- Apparent power: $|S| = \sqrt{P^2 + Q^2} \quad (\text{VA})$
- Instantaneous power: $p(t) = P(1 + \cos 2\omega t) + Q \sin 2\omega t$

Recap: Power Factor and Reactive Factor

- **Power Factor(pf):**

$$\textbf{pf} = \cos(\theta_v - \theta_i)$$

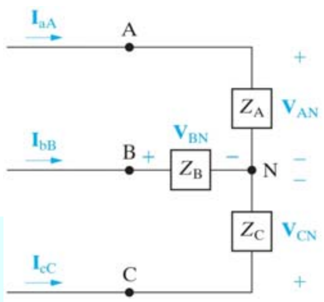
- Knowing the value of pf is not enough to find $\theta_v - \theta_i$. Problem is resolved by the following terminology:
  - ❑ **Lagging Power Factor** means current lags voltage - hence an inductive load ($\theta_v - \theta_i > 0$).
  - ❑ **Leading Power Factor** means current leads voltage - hence a capacitive load ($\theta_v - \theta_i < 0$).
- **Reactive Factor** (rf):

$$\textbf{rf} = \cos(\theta_v - \theta_i)$$

Average Power in a Balanced Y Load

- With phasors expressed using the rms values:

$$\begin{aligned} \mathcal{P}_A &= |\mathcal{V}_{AN}||\mathcal{I}_{aA}|\cos(\theta_{vA} - \theta_{iA}) \\ \mathcal{P}_B &= |\mathcal{V}_{BN}||\mathcal{I}_{bB}|\cos(\theta_{vB} - \theta_{iB}) \\ \mathcal{P}_C &= |\mathcal{V}_{CN}||\mathcal{I}_{cC}|\cos(\theta_{vC} - \theta_{iC}) \end{aligned}$$



- In a balanced 3-phase system:

$$\begin{aligned} \mathcal{V}_\phi &= |\mathcal{V}_{AN}| = |\mathcal{V}_{BN}| = |\mathcal{V}_{CN}| \\ \mathcal{I}_\phi &= |\mathcal{I}_{aA}| = |\mathcal{I}_{bB}| = |\mathcal{I}_{cC}| \\ \theta_\phi &= \theta_{vA} - \theta_{iA} = \theta_{vB} - \theta_{iB} = \theta_{vC} - \theta_{iC} \end{aligned}$$

- Total average power in terms of the rms magnitudes of the line voltage ($\mathcal{V}_L$) and line current ($\mathcal{I}_L$):

$$\mathcal{P}_T = 3\mathcal{V}_\phi \mathcal{I}_\phi \cos\theta_\phi = \sqrt{3}\mathcal{V}_L \mathcal{I}_L \cos\theta_\phi$$

Complex Power in a Balanced Y Load

- With similar derivations for the reactive power:

$$Q_T = 3 V_\phi I_\phi \sin\theta_\phi = \sqrt{3} V_L I_L \sin\theta_\phi$$

- For a balanced load:

$$S_\phi = V_{AN} I^*_{aA} = V_{BN} I^*_{bB} = V_{CN} I^*_{cC} = V_\phi I^*_\phi$$

where V_ϕ and I_ϕ represent, respectively, a voltage and a current taken from the same phase.

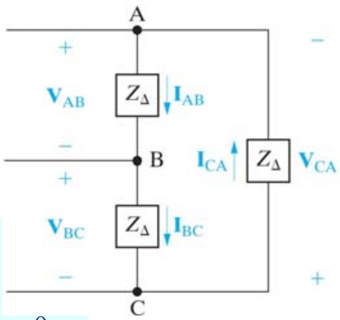
- The total complex power is given by:

$$S_T = 3S_\phi = \sqrt{3} V_L I_L e^{j\theta_\phi}$$

Power Calculations in a Balanced Δ Load

- With phasors expressed using the rms values:

$$\begin{aligned} P_A &= |V_{AB}||I_{AB}|\cos(\theta_{vAB} - \theta_{iAB}) \\ P_B &= |V_{BC}||I_{BC}|\cos(\theta_{vBC} - \theta_{iBC}) \\ P_C &= |V_{CA}||I_{CA}|\cos(\theta_{vCA} - \theta_{iCA}) \end{aligned}$$



- In a balanced 3-phase system:

$$\begin{aligned} V_\phi &= |V_{AB}| = |V_{BC}| = |V_{CA}| \\ I_\phi &= |I_{AB}| = |I_{BC}| = |I_{CA}| \\ \theta_\phi &= \theta_{vAB} - \theta_{iAB} = \theta_{vBC} - \theta_{iBC} = \theta_{vCA} - \theta_{iCA} \end{aligned}$$

- Similar calculations lead to:

$$S_T = 3S_\phi = \sqrt{3} V_L I_L e^{j\theta_\phi}$$

Instantaneous Power in 3-Phase Circuits

- With V_m and I_m being the maximum amplitudes of the phase voltage and line current, and $\theta_\phi = \theta_{vA} - \theta_{iA}$:

$$
\begin{aligned}
p_A(t) &= v_{AN} i_{aA} &&= V_m I_m \cos(\omega t)\cos(\omega t - \theta_\phi) \\
p_B(t) &= v_{BN} i_{bB} &&= V_m I_m \cos(\omega t - 120^\circ)\cos(\omega t - \theta_\phi - 120^\circ) \\
p_C(t) &= v_{CN} i_{cC} &&= V_m I_m \cos(\omega t + 120^\circ)\cos(\omega t - \theta_\phi + 120^\circ)
\end{aligned}
$$

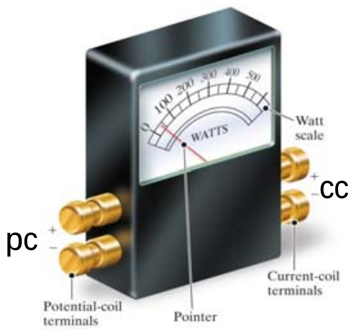
- In a balanced three-phase circuit, the total instantaneous power is invariant over time:

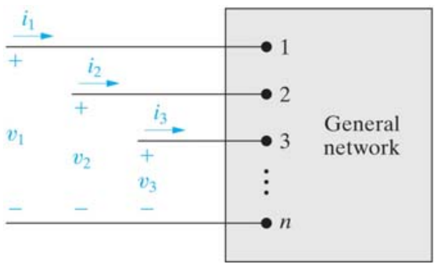
$$p_T = p_A + p_B + p_C = 1.5 V_m I_m \cos\theta_\phi$$

- Hence the torque developed at the shaft of a three-phase motor is constant $\Rightarrow$ less vibration in machinery powered by 3-phase motors.

Measuring Power in 3-Phase Circuits

Electrodynamometer





Wattmeter Reading $= |V|\ |I| \cos\theta$

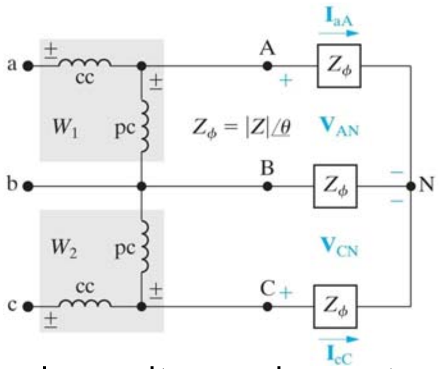
θ: Angle between V and I

n-1 wattmeters are needed for a network with n lines.

The Two-Wattmeter Method

$$\mathcal{W}_1 = |\mathcal{V}_{AB}||\mathcal{I}_{aA}|\cos\theta_1 = \mathcal{V}_L \mathcal{I}_L \cos\theta_1$$

$$\mathcal{W}_2 = |\mathcal{V}_{CB}||\mathcal{I}_{cC}|\cos\theta_2 = \mathcal{V}_L \mathcal{I}_L \cos\theta_2$$



- For an abc sequence:

$$\theta_1 = \theta_\phi + 30^\circ$$

$$\theta_2 = \theta_\phi - 30^\circ$$

θ_ϕ:phase angle between the phase voltage and current

- The total power is then given by:

$$\mathcal{P}_T = \mathcal{W}_1 + \mathcal{W}_2 = \sqrt{3}\mathcal{V}_L \mathcal{I}_L \cos\theta_\phi$$

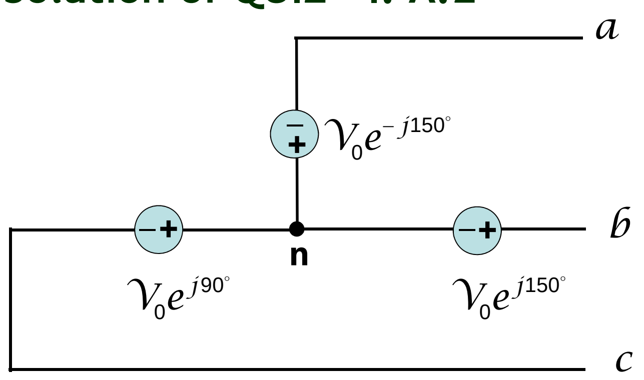
HOMEWORK - I

- Download and print the first homework assignment from Blackboard WebCT.
- Try to provide the answers on the printed pages without using extra pages. You can use both sides of the printed pages.
- Homework 1 is due on March 1, Tuesday, at the end of the class.
- Always be ready for a quiz on the due date of a homework or afterwards.

Solution of QUIZ - I: A.1

$$
\begin{aligned}
\cos\mathcal{A} + \cos\left(\mathcal{A} - 240^\circ\right) + \cos\left(\mathcal{A} + 240^\circ\right) &= \cos\mathcal{A} \\
&\quad + \cos\left(\mathcal{A}\right)\cos 240^\circ + \sin\left(\mathcal{A}\right)\sin 240^\circ \\
&\quad + \cos\left(\mathcal{A}\right)\cos 240^\circ - \sin\left(\mathcal{A}\right)\sin 240^\circ \\
&= \cos\mathcal{A}\left(1 + 2\cos 240^\circ\right) = \cos\mathcal{A}\left(1 + 2\cos\left(360^\circ - 120^\circ\right)\right) \\
&= \cos\mathcal{A}\left(1 + 2\underbrace{\cos\left(360^\circ\right)}_{1}\cos\left(120^\circ\right) + 2\underbrace{\sin\left(360^\circ\right)}_{0}\sin\left(120^\circ\right)\right) \\
&= \cos\mathcal{A}\left(1 + 2\cos\left(90^\circ + 30^\circ\right)\right) \\
&= \cos\mathcal{A}\left(1 + 2\underbrace{\cos\left(90^\circ\right)}_{0}\cos\left(30^\circ\right) - 2\underbrace{\sin\left(90^\circ\right)}_{1}\underbrace{\sin\left(30^\circ\right)}_{1/2}\right) \\
&= 0
\end{aligned}
$$

Solution of QUIZ - I: A.2



$$
\begin{aligned}
V_a &= -V_0e^{-j150^\circ} &&= V_0e^{j30^\circ} \\
V_b &= +V_0e^{j150^\circ} &&= V_0e^{j150^\circ} \\
V_c &= -V_0e^{j90^\circ} &&= V_0e^{-j90^\circ}
\end{aligned}
$$

Balanced 3-phase voltage source with acb (negative) phase sequence