7.1 Power Density Spectrum

Also called "Power Spectral Density"

Deterministic Signal $X(t)$

- Let $x(t)$ be a deterministic signal
  $\Rightarrow$ the spectral properties are contained in the Fourier Transform
  $$X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$
- $X(w)$ is called the spectrum of $x(t)$ and has the unit of volts per hertz.
- Thus, $X(w)$ is the voltage density spectrum.
- $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jwt} dw$$

Random Process $X(t)$

- $X(t)$ is a collection of sample function $x(t)$.
- $X(w)$ may not exist for all $X(t)$.

$\Rightarrow$ solution: take the power density spectrum.

Power Density Spectrum

For a random process $X(t)$,

Define $x_T(t) = \begin{cases} x(t) & -T<t<T \\ 0 & \text{elsewhere} \end{cases}$

then $\int_{-T}^{T} |x_T(t)| dt < \infty$

$\Rightarrow$ Fourier Transform exist

$$\Rightarrow X_T(w) = \int_{-T}^{T} x_T(t) e^{-jwt} dt$$
$$= \int_{-T}^{T} x(t) e^{-jwt} dt$$

* The Energy in $x(t)$ in the interval $(-T, T)$

$$E(T) = \int_{-T}^{T} x_T^2(t) dt$$
$$= \int_{-T}^{T} x^2(t) dt$$

* Using Parsevals Theorem

$$E(T) = \int_{-T}^{T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(w)|^2 dw$$

— The power $P(T)$ is

$$P(T) = \frac{1}{2T} \int_{-T}^{T} x^2(t) dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(w)|^2}{2T} dw$$

Since $P(T)$ is a Random Variable, the average power is

$$P_{XX} = E\{P(T)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\{X^2(t)\}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\{|X_T(w)|^2\}}{2T}\,dw$$

Note that:

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\,dt = \underbrace{A\{E\{X^2(t)\}\}}_{\text{Time Average of second moment}}$$

– For a wide-sense stationary Random Process

$$P_{XX} = E\{X^2(t)\} = \overline{X^2} \equiv \text{a constant}$$

Also:

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\{|X_T(w)|^2\}}{2T}\,dw$$

Define $\underbrace{\varphi_{XX}(w) = \lim_{T\to\infty} \frac{E\{|X_T(w)|^2\}}{2T}}_{\text{Power Density Spectrum}}$

$\Rightarrow \cancel{P_{XX} =}$

$$\Rightarrow P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{XX}(w)\,dw$$

where $\varphi_{XX}(w) \equiv$ Power Density Spectrum

Example

$$X(t) = A_0\cos(w_0 t + \Theta)$$

where A_0 and w_0 are real constants and Θ is a uniform R.V $(0, \frac{\pi}{2})$

(a) Find the Power Density Spectrum

(b) Find the average power P_{XX}

(a) Power Density Spectrum

$$\varphi_{XX}(w) = \lim_{T\to\infty} \frac{E[|X_T(w)|^2]}{2T}$$

$$X_T(w) = \int_{-T}^{T} A_0\cos(w_0 t + \theta)\, e^{-jwt}\,dt$$

$$= \int_{-T}^{T} \frac{A_0}{2}\left[e^{jw_0 t+\theta} + e^{-jw_0 t-\theta}\right] e^{-jwt}\,dt$$

$$= \frac{A_0}{2} e^{j\theta}\int_{-T}^{T} e^{j(w_0-w)t}\,dt + \frac{A_0}{2} e^{-j\theta}\int_{-T}^{T} e^{-j(w_0+w)t}\,dt$$

$$= \frac{A_0}{2} e^{j\theta}\left[\frac{e^{j(w_0-w)t}}{j(w_0-w)}\right]_{-T}^{T} + \frac{A_0}{2} e^{-j\theta}\left[\frac{e^{-j(w_0+w)t}}{-j(w_0+w)}\right]_{-T}^{T}$$

$$= \frac{A_0}{2} e^{j\theta}\left[\frac{e^{j(w_0-w)T} - e^{-j(w_0-w)T}}{j(w_0-w)}\right] + \frac{A_0}{2} e^{-j\theta}\left[\frac{e^{-j(w_0+w)T} - e^{j(w_0+w)T}}{-j(w_0+w)}\right]$$

$$= A_0 T e^{j\theta} \frac{\sin\{(w-w_0)T\}}{(w-w_0)T} + A_0 T e^{-j\theta} \frac{\sin\{(w+w_0)T\}}{(w+w_0)T}$$

$$|X_T(w)|^2 = X_T(w)\, X_T^*(w)$$

$$= \left[A_0 T e^{j\theta} \frac{\sin\{(w-w_0)T\}}{(w-w_0)T} + A_0 T e^{-j\theta} \frac{\sin[(w+w_0)T]}{(w+w_0)T}\right] \cdot \left[A_0 T e^{-j\theta} \frac{\sin((w-w_0)T)}{(w-w_0)T} + A_0 T e^{j\theta} \frac{\sin\{(w+w_0)T\}}{(w+w_0)T}\right]$$

$$= A_0^2 T^2 \frac{\sin^2\{(w-w_0)T\}}{\{(w-w_0)T\}^2} + A_0^2 T^2 e^{2j\theta} \frac{\sin\{(w-w_0)T\}}{(w-w_0)T} \frac{\sin\{(w+w_0)T\}}{(w+w_0)T} + A_0^2 T^2 e^{-2j\theta} \frac{\sin\{(w+w_0)T\}}{(w+w_0)T} \frac{\sin\{(w-w_0)T\}}{(w-w_0)T} + A_0^2 T^2 \frac{\sin^2\{(w+w_0)T\}}{\{(w+w_0)T\}^2}$$

Ignoring the cross terms since they result in very small values, we get

$$\frac{E\{|X_T(w)|^2\}}{2T} = \frac{A_0^2 \pi}{2}\left\{\frac{T}{\pi} \frac{\sin^2\{(w-w_0)T\}}{[(w-w_0)T]^2} + \frac{T}{\pi} \frac{\sin^2\{(w+w_0)T\}}{[(w+w_0)T]^2}\right\}$$

Since $\lim_{T\to\infty} \frac{T}{\pi}\left[\frac{\sin(\alpha T)}{\alpha T}\right]^2 = \delta(\alpha)$

$$\Rightarrow \varphi_{XX}(w) = \frac{A_0^2 \pi}{2}\left[\delta(w-w_0) + \delta(w+w_0)\right]$$

(b) The average power

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A_0^2 \pi}{2}\{\delta(w-w_0) + \delta(w+w_0)\}\,dw$$

$$= \frac{A_0^2}{2}$$

Properties of the Power Density Spectrum

① $\Phi_{XX}(\omega) \geq 0$

② $\Phi_{XX}(-\omega) = \Phi_{XX}(\omega)$

③ $\Phi_{XX}(\omega)$ is real

④ $\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{XX}(\omega)\, d\omega = A\{E[X^2(t)]\}$

⑤ $\Phi_{\dot{X}\dot{X}}(\omega) = \omega^2 \Phi_{XX}(\omega)$

⑥ $\Phi_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t, t+\tau)] e^{-j\omega\tau}\, d\tau$

and $\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{j\omega\tau}\, d\omega = A\{R_{XX}(t, t+\tau)\}$

Thus, $\Phi_{XX}(\omega) \xleftrightarrow{F.T} A[R_{XX}(t, t+\tau)]$

For wide-Sense Stationary

$$\Phi_{XX}(\omega) \xleftrightarrow{F.T} R_{XX}(\tau)$$

7.2 Relation Between Power Spectrum and Autocorrelation Function.

$$R_{XX}(\tau) \xleftrightarrow{F.T} \Phi_{XX}(\omega)$$

$$\Rightarrow \Phi_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{j\omega\tau}\, d\omega$$

Example 7.2-1

$R_{XX}(\tau) = \frac{A_0^2}{2}\cos(\omega_0\tau)$ is the Autocorrelation of the Random Process

$$X(t) = A\cos(\omega_0 t + \theta)$$

The power spectral Density

$$\Phi_{XX}(\omega) = F.T\{R_{XX}(\tau)\}$$

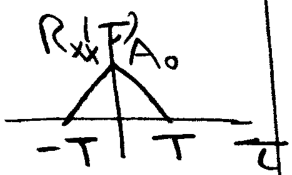
Using Appendix D.

$$\Phi_{XX}(\omega) = \frac{A_0^2 \pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

Example 7.2

$$R_{xx}(\tau) = \begin{cases} A_0\left[1 - \frac{|\tau|}{T}\right] & -T \le \tau \le T \\ 0 & \text{elsewhere} \end{cases}$$

$$= A_0 \, \text{tri}\left(\frac{\tau}{T}\right)$$



$$\Rightarrow \varphi_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$= A_0 \int_{-T}^{0} \left[1 + \frac{\tau}{T}\right] e^{-j\omega\tau} d\tau$$

$$+ A_0 \int_{0}^{T} \left[1 - \left(\frac{\tau}{T}\right)\right] e^{-j\omega\tau} d\tau$$

$$= A_0 T \, \text{sinc}^2\left(\frac{\omega T}{2}\right)$$

Sinc Funtion

~~$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$~~

$$\text{Sinc}(x) = \frac{\sin x}{x}$$