Chapter 13 Cover

Rate Distortion Theory

- Representing real numbers with finite symboles will cause distortion.
- We can define a distortion measure.
- For a given source and distortion measure, what is the minimum expected distortion achievable at a particular rate?
- What is the minimum rate description required to achieve a particular distortion?

13.1 Quantization

- Since a continuous random source requires infinite precision to represent exactly, we cannot reproduce it exactly using a finite rate code.

  The question is then to find the best possible representation for any given rate.

- Def:

  $X \equiv$ the random variable.

  $\hat{X}(X) \equiv$ representation of X.

- Assume we want to use R bits to represent X

  $\Rightarrow \hat{X}$ can take 2^R values

- The design problem is to find the optimum set of values for $\hat{X}$ (called the reproduction points or codepoints) and the regions that are associated with each value $\hat{X}$.

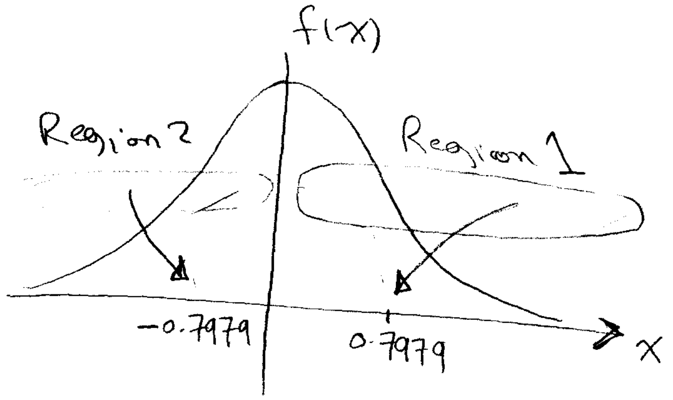
Example:

Let $X \sim \mathcal{N}(0, \sigma^2)$, and assume a squared error distortion measure. In this case, we wish to find the function $\hat{X}(X)$ such that $\hat{X}$ takes on at most 2^R values and minimizes $E[X-\hat{X}(X)]^2$.

- Let $R = 1$ bit

$$\Rightarrow \hat{X}(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\,\sigma, & \text{if } x \geq 0 \\ -\sqrt{\frac{2}{\pi}}\,\sigma, & \text{if } x < 0 \end{cases}$$

- The above assignment minimize the squared error.



- For any R bits representation, the assignment should look for the optimal regions that ~~minimizes~~ minimize distortion.

- Important Result:

  Vector quantization minimizes distortion more than a single random variable quantization.

13.2: Definitions

Source: Produces an i.i.d sequence of n random variable $X_1, X_2, \dots X_n$ each with $p(x)$ pdf and $x \in \mathcal{X}$.

Encoder: Mapps the source sequence X^n to a quantized code with an index
$$f_n(X^n) \in \{1, 2, \dots, 2^{nR}\}$$

Decoder: represents X^n by an estimate $\hat{X}^n \in \hat{\mathcal{X}}^n$.

$$X^n \rightarrow \boxed{\text{Encoder}} \xrightarrow{f_n(X^n) \in \{1,2,\dots,2^{nR}\}} \boxed{\text{Decoder}} \rightarrow \hat{X}^n$$

Def: Distortion Function

is a mapping
$$d: \mathcal{X} \times \hat{\mathcal{X}} \rightarrow R^+$$
where
$\mathcal{X} \equiv$ Source alphabet
$\hat{\mathcal{X}} \equiv$ reproduction alphabet
$R^+ \equiv$ non-negative real numbers
$d(x, \hat{x}) \equiv$ the distortion between x and $\hat{x}$.

Def. Bounded distortion

if the maximum value of the distortion is finite
$$d_{max} \overset{def}{=} \max_{x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}} d(x, \hat{x}) < \infty$$

Example of distortion functions:

- Hamming distortion:

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x=\hat{x} \\ 1 & \text{if } x\neq\hat{x} \end{cases}$$

- Squared error distortion:

$$d(x,\hat{x}) = (x-\hat{x})^2$$

Def: Average distortion between two sequences x^n and $\hat{x}^n$

$$d(x^n,\hat{x}^n) = \frac{1}{n}\sum_{i=1}^{n} d(x_i,\hat{x}_i)$$

Def: A $(2^{nR}, n)$ rate distortion code consists of an encoding function:

$$f_n : \mathcal{X}^n \longrightarrow \{1,2,\dots,2^{nR}\}$$

and a decoding (reproduction) function:

$$g_n : \{1,2,\dots,2^{nR}\} \longrightarrow \mathcal{X}^n$$

The distortion associated with the $(2^{nR}, n)$ code is:

$$D = E\left\{d(X^n, \underbrace{g_n(f_n(X^n))}_{\hat{X}^n})\right\}$$

where the expectation is with respect to the prob. dist. on X;

$$\Rightarrow D = \sum_{x^n} p(x^n)\, d(x^n, g_n(f_n(x^n)))$$

note that:

$$g_n(1), g_n(2), \dots, g_n(2^{nR}) \equiv \hat{X}^n(1), \dots, \hat{X}^n(2^{nR}) \equiv \text{codebook}$$

Also,

$$f_n^{-1}(1), \dots, f_n^{-1}(2^{nR}) \equiv \text{assignment regions}$$

Def: A rate distortion pair (R,D) is said to be achievable if there exists a sequence of $(2^{nR}, n)$ rate distortion codes (f_n, g_n) with

$$\lim_{n\to\infty} E\{d(X^n, g_n(f_n(X^n)))\} \leq D$$

Def: The rate distortion region for a source is the closure of the set of achievable rate distortion pairs (R, D)

Def: The rate distortion function $R(D)$ is the infimum of rates R such that (R, D) is in the rate distortion region of the source given D.

Def: The distortion rate function $D(R)$ is the infimum of all distortions D such that (R, D) is in the rate distortion region of the source for a given rate R.

Theorem 13.2.1:

The minimum achievable rate at distortion D for an i.i.d source X with distribution $p(x)$ and bounded distortion function $d(x, \hat{x})$ is equal to the associated information rate distortion function. Thus,

$$R(D) = \min_{p(\hat{x}|x): \sum_{(x,\hat{x})} p(x) p(\hat{x}|x) d(x,\hat{x}) \leq D} I(X; \hat{X})$$

where the minimization is over all conditional distributions $p(\hat{x}|x)$ for which the joint distribution $p(x,\hat{x}) = p(x) p(\hat{x}|x)$ satisfies the expected distortion constraint.

13.3.2 Gaussian Source

Theorem 13.3.2:

The rate distortion function for a $N(0, \sigma^2)$ source with squared error distortion is:

$$R(D) = \begin{cases} \frac{1}{2} \log \frac{\sigma^2}{D}, & 0 \leq D \leq \sigma^2 \\ 0, & D > \sigma^2 \end{cases}$$

Proof: Let $X \sim N(0, \sigma^2)$

the rate distortion function is

$$R(D) = \min_{f(\hat{x}|x): E(\hat{X}-X)^2 \leq D} I(X; \hat{X})$$

$$\begin{aligned}
I(X; \hat{X}) &= h(X) - h(X|\hat{X}) \\
&= \tfrac{1}{2} \log(2\pi e)\sigma^2 - h(X - \hat{X}|\hat{X}) \\
&\geq \tfrac{1}{2} \log(2\pi e)\sigma^2 - h(X - \hat{X}) \\
&\geq \tfrac{1}{2} \log(2\pi e)\sigma^2 - h(N(0, E(X-\hat{X})^2)) \\
&= \tfrac{1}{2} \log(2\pi e)\sigma^2 - \tfrac{1}{2} \log(2\pi e) E(X-\hat{X})^2 \\
&\geq \tfrac{1}{2} \log(2\pi e)\sigma^2 - \tfrac{1}{2} \log(2\pi e) D \\
&= \tfrac{1}{2} \log \frac{\sigma^2}{D}
\end{aligned}$$

$$\Rightarrow R(D) \geq \tfrac{1}{2} \log \frac{\sigma^2}{\quad}$$

- We can express the distortion in terms of the rate as:

$$D(R) = \sigma^2 \, 2^{-2R}$$

- Each bit reduces the expected distortion by a factor of 4.