Image Enhancement in the Frequency Domain
Part I

Dr. Samir H. Abdul-Jauwad
Electrical Engineering Department
King Fahd University of Petroleum & Minerals

Fundamentals

- Fourier: a periodic function can be represented by the sum of sines/cosines of different frequencies,multiplied by a different coefficient (Fourier series)
- Non-periodic functions can also be represented as the integral of sines/cosines multiplied by weighing function (Fourier transform)

Introduction to the Fourier Transform

- f(x): continuous function of a real variable x

- Fourier transform of f(x):

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx \qquad \text{Eq. 1}$$

where $j = \sqrt{-1}$

Introduction to the Fourier Transform

- (u) is the frequency variable.
- The integral of Eq. 1 shows that F(u) is composed of an infinite sum of sine and cosine terms and…
- Each value of u determines the frequency of its corresponding sine-cosine pair.

Introduction to the Fourier Transform

- Given F(u), f(x) can be obtained by the inverse Fourier transform:

$$\Im^{-1}\{F(u)\} = f(x)$$
$$= \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

- The above two equations are the Fourier transform pair.

Introduction to the Fourier Transform

- Fourier transform pair for a function f(x,y) of two variables:

$$\Im\{f(x,y)\} = F(u,v) = \int \int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

and

$$\Im^{-1}\{F(u,v)\} = f(x,y) = \int \int_{-\infty}^{\infty} F(u,v)\exp[j2\pi(ux+vy)]dudv$$

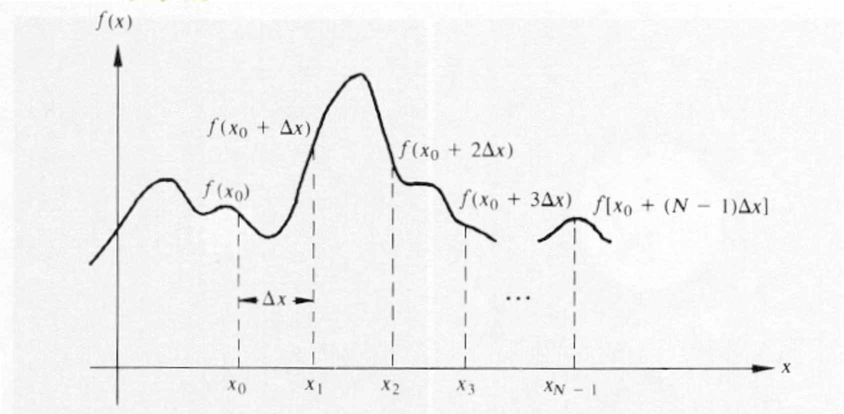
where u,v are the frequency variables.

Discrete Fourier Transform

- A continuous function f(x) is discretized into a sequence:

$$\{ f(x_0), f(x_0 + \Delta x), f(x_0 + 2\Delta x), \ldots, f(x_0 + [N-1]\Delta x) \}$$

by taking N or M samples Δx units apart.



Discrete Fourier Transform

- Where x assumes the discrete values (0,1,2,3,…,M-1) then

$$f(x) = f(x_0 + x\Delta x)$$

- The sequence {f(0),f(1),f(2),...f(M-1)} denotes any M uniformly spaced samples from a corresponding continuous function.

Discrete Fourier Transform

- The discrete Fourier transform pair that applies to sampled functions is given by:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\exp[-j2\pi ux/M]$$ For u=0,1,2,...,M-1

and

$$f(x) = \sum_{u=0}^{M-1} f(u)\exp[j2\pi ux/M]$$ For x=0,1,2,...,M-1

Discrete Fourier Transform

- To compute F(u) we substitute u=0 in the exponential term and sum for all values of x
- We repeat for all M values of u
- It takes M*M summations and multiplications

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\exp[-j2\pi ux/M]$$ For u=0,1,2,…,M-1

- The Fourier transform and its inverse always exist!

Discrete Fourier Transform

- The values u = 0, 1, 2, …, M-1 correspond to samples of the continuous transform at values 0, Δu, $2\Delta u$, …, $(M-1)\Delta u$.

- i.e. F(u) represents $F(u\Delta u)$, where:

$$\Delta u = \frac{1}{M\Delta x}$$

Details

$$e^{j\theta} = \cos\theta = j\sin\theta$$

$$\cos(-\theta) = \cos(\theta)$$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux / M - j\sin 2\pi ux / M]$$

- Each term of the FT (F(u) for every u) is composed of the sum of all values of f(x)

Introduction to the Fourier Transform

- The Fourier transform of a real function is generally complex and we use polar coordinates:

$$F(u) = R(u) + jI(u)$$

$$F(u) = |F(u)|e^{j\phi(u)}$$

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

Introduction to the Fourier Transform

- |F(u)| (magnitude function) is the Fourier spectrum of f(x) and ϕ(u) its phase angle.

- The square of the spectrum

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

is referred to as the power spectrum of f(x) (spectral density).

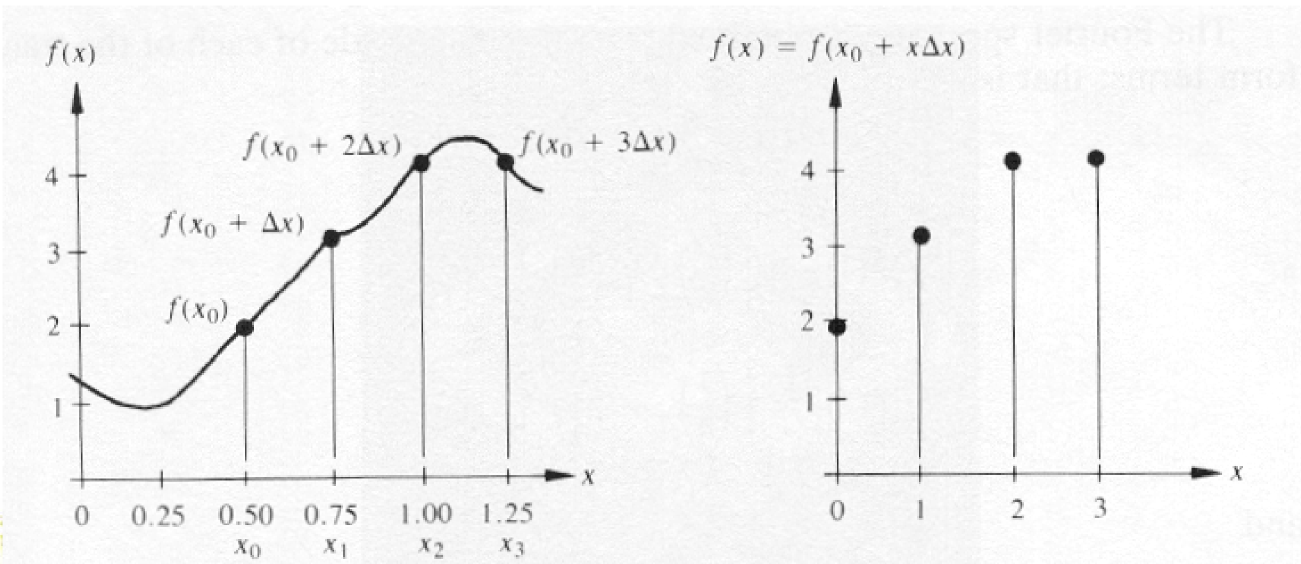
Introduction to the Fourier Transform

- Fourier spectrum: $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$

  - Phase: $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$

- Power spectrum: $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$

Discrete Fourier Transform



Discrete Fourier Transform

- In a 2-variable case, the discrete FT pair is:

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M+vy/N)]$$

For u=0,1,2,…,M-1 and v=0,1,2,…,N-1

AND:

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M+vy/N)]$$

For x=0,1,2,…,M-1 and y=0,1,2,…,N-1

Discrete Fourier Transform

- Sampling of a continuous function is now in a 2-D grid (Δx, Δy divisions).
- The discrete function f(x,y) represents samples of the function $f(x_0+x\Delta x, y_0+y\Delta y)$ for x=0,1,2,…,M-1 and y=0,1,2,…,N-1.

$$\Delta u = \frac{1}{M\Delta x}, \qquad \Delta v = \frac{1}{N\Delta y}$$

Discrete Fourier Transform

- When images are sampled in a square array, M=N and the FT pair becomes:

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux+vy)/N]$$

For u,v=0,1,2,…,N-1

AND:

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux+vy)/N]$$

For x,y=0,1,2,…,N-1

Some Properties of the 2-D Fourier Transform

Translation

Distributivity and Scaling

Rotation

Periodicity and Conjugate Symmetry

Separability

Convolution and Correlation

Translation

$$f(x,y)\exp[j2\pi(u_0x/M+v_0y/N)]\Leftrightarrow F(u-u_0,v-v_0)$$

and

$$f(x-x_0,y-y_0)\Leftrightarrow F(u,v)\exp[-j2\pi(ux_0/M+vy_0/N)]$$

Translation

- The previous equations mean:
  - Multiplying f(x,y) by the indicated exponential term and taking the transform of the product results in a shift of the origin of the frequency plane to the point (u_0,v_0).
  - Multiplying F(u,v) by the exponential term shown and taking the inverse transform moves the origin of the spatial plane to (x_0,y_0).
  - A shift in f(x,y) doesn't affect the magnitude of its Fourier transform

Distributivity and Scaling

$$\Im\{ f_1(x,y) + f_2(x,y)\} = \Im\{ f_1(x,y)\} + \Im\{ f_2(x,y)\}$$

$$\Im\{ f_1(x,y) \cdot f_2(x,y)\} \neq \Im\{ f_1(x,y)\} \cdot \Im\{ f_2(x,y)\}$$

- Distributive over addition but not over multiplication.

Distributivity and Scaling

- For two scalars a and b,

$$af(x,y) \Leftrightarrow aF(u,v)$$

$$f(ax,by) \Leftrightarrow \frac{1}{|ab|}F(u/a,v/b)$$

Rotation

- Polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad u = \omega\cos\varphi, \quad v = \omega\cos\varphi$$

Which means that:

$$f(x, y), F(u, v) \text{ become } f(r, \theta), F(\omega, \varphi)$$

Rotation

$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$$

- Which means that rotating f(x,y) by an angle θ_0 rotates F(u,v) by the same angle (and vice versa).

Periodicity & Conjugate Symmetry

- The discrete FT and its inverse are periodic with period N:

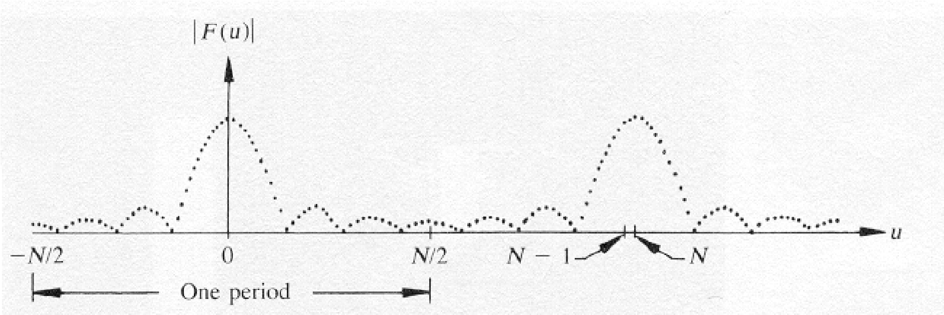
$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

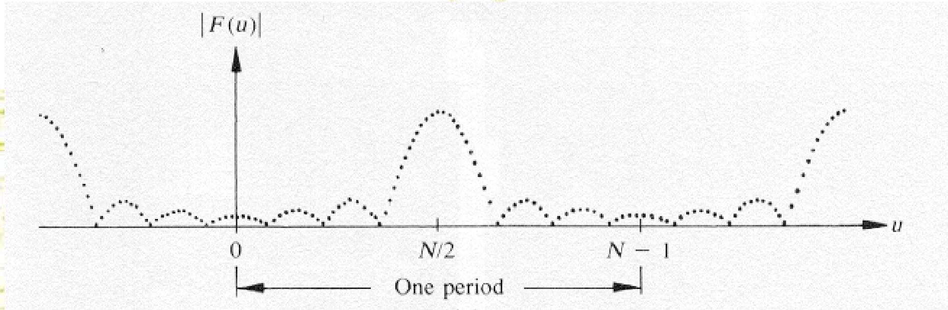
Periodicity & Conjugate Symmetry

- Although F(u,v) repeats itself for infinitely many values of u and v, only the M,N values of each variable in any one period are required to obtain f(x,y) from F(u,v).

- This means that only one period of the transform is necessary to specify F(u,v) completely in the frequency domain (and similarly f(x,y) in the spatial domain).

Periodicity & Conjugate Symmetry





(shifted spectrum) move the origin of the transform to u=N/2.

Periodicity & Conjugate Symmetry

- For real f(x,y), FT also exhibits conjugate symmetry:

$$F(u,v) = F^*(-u,-v)$$

or $$|F(u,v)| = |F(-u,-v)|$$

Periodicity & Conjugate Symmetry

- In essence:

$$F(u) = F(u+N)$$

$$|F(u)| = |F(-u)|$$

- i.e. F(u) has a period of length N and the magnitude of the transform is centered on the origin.

Separability

- The discrete FT pair can be expressed in separable forms which (after some manipulations) can be expressed as:

$$F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)\exp[-j2\pi ux/M]$$

Where:

$$F(x,v) = \left[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]\right]$$

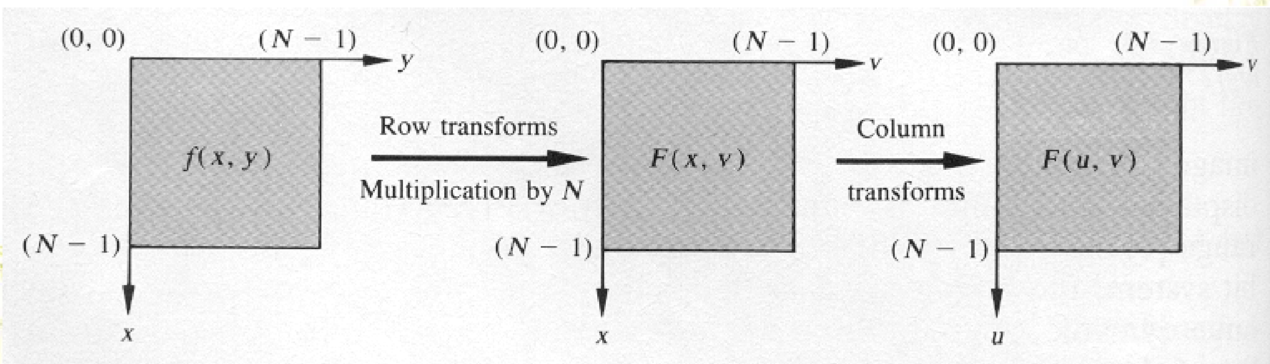
Separability

- For each value of x, the expression inside the brackets is a 1-D transform, with frequency values v=0,1,…,N-1.

- Thus, the 2-D function F(x,v) is obtained by taking a transform along each row of f(x,y) and multiplying the result by N.

Separability

- The desired result F(u,v) is then obtained by making a transform along each column of F(x,v).



Convolution

- Convolution theorem with FT pair:

$$f(x) * g(x) \Leftrightarrow F(u)G(u)$$

$$f(x)g(x) \Leftrightarrow F(u) * G(u)$$

Convolution

- Discrete equivalent:

$$f_e(x) * g_e(x) = \frac{1}{M}\sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

•Discrete, periodic array of length M.
•x=0,1,2,…,M-1 describes a full period of $f_e(x)*g_e(x)$.
•Summation replaces integration.

Correlation

- Correlation of two functions: f(x) $\circ$ g(x)

$$f(x) \circ g(x) = \int_{-\infty}^{\infty} f^*(\alpha) g(x+\alpha) d\alpha$$

- Types: autocorrelation, cross-correlation
  - Used in template matching

Correlation

- Correlation theorem with FT pair:

$$f(x,y)\circ g(x,y)\Leftrightarrow F^*(u,v)G(u,v)$$

$$f^*(x,y)g(x,y)\Leftrightarrow F(u,v)\circ G(u,v)$$

Correlation

- Discrete equivalent:

$$f_e(x) \circ g_e(x) = \frac{1}{M} \sum_{m=0}^{M-1} f_e{}^*(m) g_e(x+m)$$

For x=0,1,2,…,M-1

Fast Fourier Transform

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \exp[-j2\pi ux / M]$$

- Number of complex multiplications and additions to implement Fourier Transform: M^2 (M complex multiplications and N-1 additions for each of the N values of u).

Fast Fourier Transform

- The decomposition of FT makes the number of multiplications and additions proportional to M $\log_2 M$:
  - Fast Fourier Transform or FFT algorithm.
- E.g. if M=1021 the usual method will require 1000000 operations, while FFT will require 10000.

Fast Fourier Transform

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Questions?