Introduction to Discrete-Time Signals and Systems

The z-Transform

Lecture #39

The material to be covered in this lecture is as follows:

- Introduction to the z-transform
- Definition of the z-transform
- Derivation of the z-transform
- Region of convergence for the transform
- Examples.

After finishing this lecture you should be able to:

- Find the *z*-transform for a given signal utilizing the *z*-transform definition
- Calculate the region of convergence for the transform

Derivation of the *z*-Transform

- The *z*-transform is the basic tool for the analysis and synthesis of discrete-time systems.
- The *z*-transform is defined as follows:

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$$

- The coefficient $x(nT)$ denote the sample value and z^{-n} denotes that the sample occurs n sample periods after the t=0 reference.
- Rather than starting form the given definition for the *z*-transform, we may start from the continuous-time function and derive the *z*-transform. This is done in the next slide.

Derivation of the *z*-transform

The sampled signal may be written as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$

Since $\delta(t-nT)=0$ for all t except at $t=nT$, $x(t)$ can be replaced by $x(nT)$.
Assuming $x(t)=0$ for $t<0$. Then,

$$x_s(t) = \sum_{n=0}^{\infty} x(nt)\delta(t-nT)$$

Taking Laplace transform yields

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nt)\delta(t-nT)e^{-st}dt$$

Rearranging

$$X_s(s) = \sum_{n=0}^{\infty} x(nT)\int_0^{\infty} \delta(t-nT)e^{-st}dt$$

By sifting property of the delta function

$$X_s(s) = \sum_{n=0}^{\infty} x(nT)e^{-snT}$$

Continue Derivation…

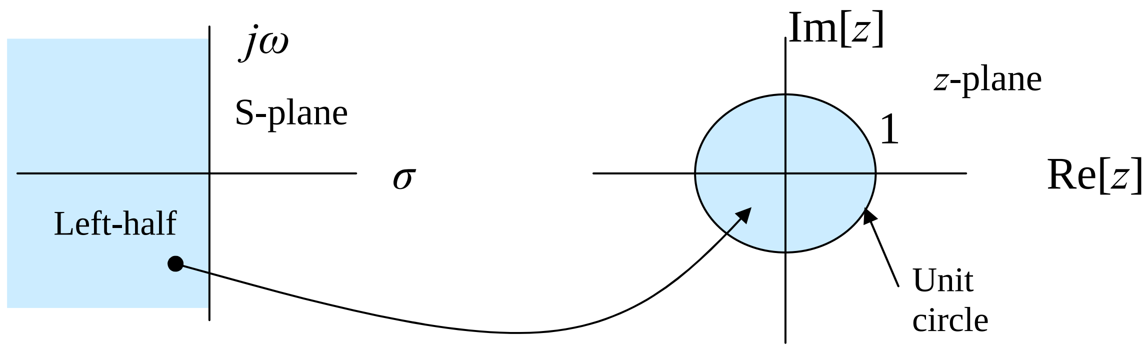
Defining the complex variable z as the Laplace time-shift operator

$$z = e^{sT}$$

Hence,

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$$

We could have started from here but it is good to relate to the s-domain



In the s-domain the left-half plane corresponds to $\sigma < 0$ is mapped to $|z|<1$ in the z-plane which is the region inside the unit circle.

Region of Convergence (ROC)

$|z|$ is converged for σ <0 (left-half of s-plane). This corresponds to $|z|$<1. This is the region inside the unit circle.

$|z|$ is NOT converged for σ > 0 (right-half of s-plane). This corresponds to $|z|$>1 which is the region outside the unit circle

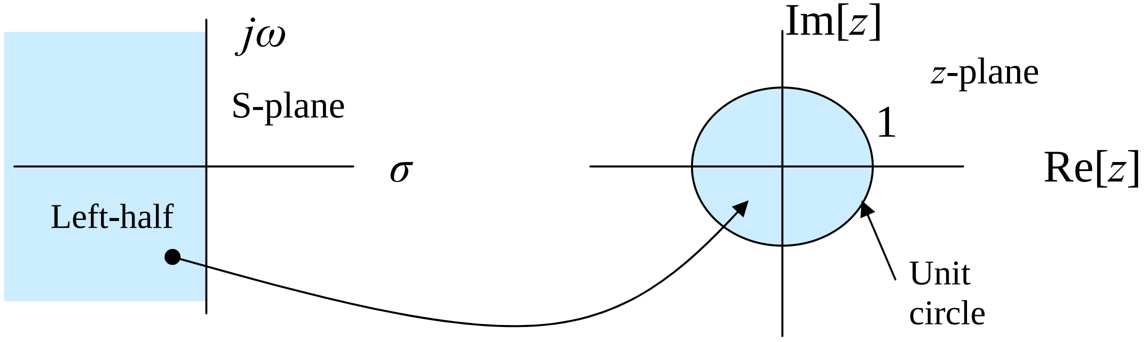
$$z = e^{sT}$$

$$s = \sigma + j\omega$$

$$z = e^{\sigma T} e^{j\omega T}$$

$$|z| = e^{\sigma T}$$

The mapping of the Laplace variable s into the z-plane through z=e^{sT} is illustrated in the figure below:



The Z-Transform in Summary

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n} \quad where \quad z = e^{sT}$$

- The coefficient *x(nT)* denotes the sampled value
- z^{-n} denotes that the sample occurs *n* sample periods after the t=0 reference.
- e^{sT} is simply the T-second time shift
- The parameter z is simply shorthand notation for the Laplace time shift operator
- For instance, $30z^{-40}$ denotes a sample, having value 30, which occurs *40* sample periods after the t=0 reference
- The definition of *z*-transform can also be written as: (other text books)

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad for \quad n \geq 0$$

  where the square bracket is used to indicate discrete times.
- It worth to mention that Matlab has special tools for Z-transform.

Example 39.1

Determine the *z*-transform for the following signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & otherwise \end{cases}$$

- Solution:

We know that

$$\mathrm{X}(\mathrm{z}) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

hence

$$\mathrm{X}(\mathrm{z}) = \sum_{-1}^{2} x[n] z^{-n} = x[-1] z^{-(-1)} + x[0] z^{-(0)} + x[1] z^{-(1)} + x[2] z^{-(2)}$$

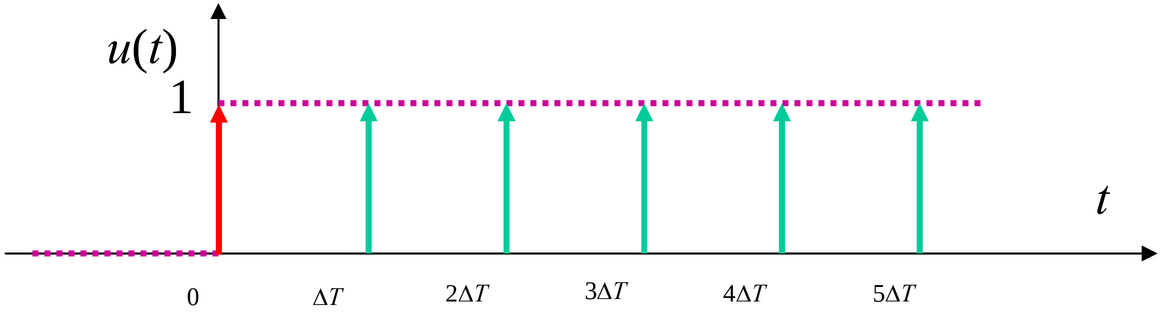
$$X(z) = z + 2 - z^{-1} + z^{-2}$$

Example 39.2: Sampled Step Function (*Important Functions*)

Consider a unit step sample sequence defined by

$$x[n]=1,\ n\geq 0$$

Find the z-transform.



Solution:

$$U(z)=X(z)=1+z^{-1}+z^{-2}+z^{-3}+..........=\sum_{n=0}^{\infty} z^{-n}=\frac{1}{1-z^{-1}}, \qquad |z|>1$$

The sum converges absolutely to $1/(1-z^{-1})$ outside the unit circle $|z|>1$

$$X(z)=\sum_{n=0}^{\infty} z^{-n}$$

Sampled Dirac Delta Function (*an other important function*)

The Dirac Delta Function is defined to be

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

For a delayed version of delta is defined as

$$\delta(n-k) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

Applying the definition of the *z*-transform

Dirac function

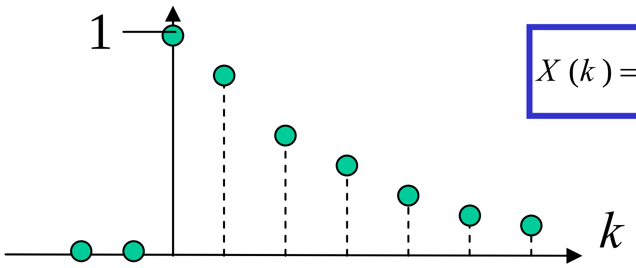
$$X(z) = \sum_{k=0}^{\infty} \delta(t) z^{-s\Delta T} = \delta(0) = 1$$

$$X(z) = 1$$

The Unit Exponential Sequence

The unit exponential sequence is defined to be

$$x(k) = \begin{cases} e^{-\alpha k} & k, \alpha > 0 \\ 0 & k < 0 \end{cases}$$



$$X(k) = \frac{1}{1 - e^{-\alpha} z^{-1}} = \frac{z}{z - e^{-\alpha}}$$

Apply z-transform definition $X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$ we get

$$X(z) = \sum_{k=0}^{\infty} e^{-\alpha k} z^{-k} = \sum_{k=0}^{\infty} (e^{-\alpha} z^{-1})^{k}$$

$$X(k) = \frac{1}{1 - e^{-\alpha} z^{-1}} = \frac{z}{z - e^{-\alpha}}$$

where $|z| > e^{-\alpha T}$

if $k = e^{-\alpha T}$ then

$$X(z) = \frac{1}{1 - k z^{-1}} = \frac{z}{z - k}$$

Example 39.3

Determine the z-transform of the signal

$$x[n] = 0.5^n u[n]$$

Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane

Solution:

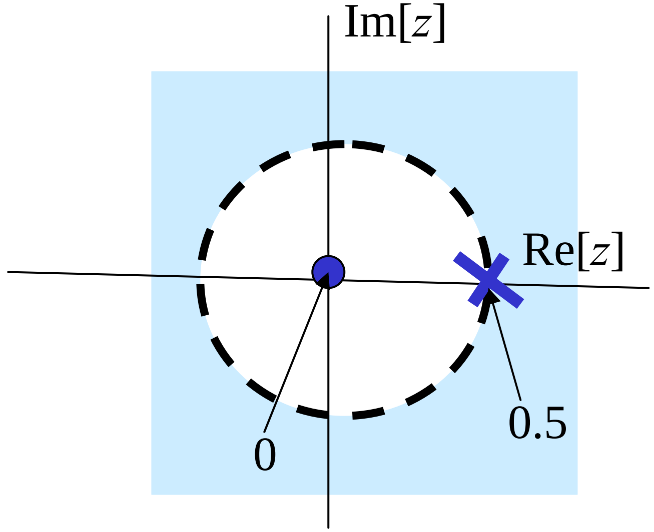
Substituting is the definition of the z-transform

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n$$

This is a geometric series of infinite length in the ratio $0.5/z$; the sum converges, provided that $|0.5/z| < 1$ or $|z| > 0.5$. Hence the z-transform is

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n = \frac{1}{1-0.5z^{-1}}, \qquad |z| > 0.5$$

$$= \frac{z}{z-0.5}, \qquad |z| > 0.5$$



Pole at z=0, zero at z=0.5, ROC is the light blue region

Self Test:

1) Determine the z-transform for the following signal

$$x[n] = \left\{ \left(\frac{1}{2}\right)^n u[n] \right.$$

Solution:

- Utilizing the definition of the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Hence

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] \quad z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \quad z^{-n}$$

$$X(z) = \left(\frac{1}{2}\right)^0 \quad z^{-0} + \left(\frac{1}{2}\right)^1 \quad z^{-1} + \left(\frac{1}{2}\right)^2 \quad z^{-2} + \ldots$$

$$X(z) = 1 + \left(\frac{1}{2z}\right) + \left(\frac{1}{4z^2}\right) + \ldots$$

2) Find the *z*- transform of the following signal:

$$X(nT) = a^n \cos\left(\frac{n\pi}{2}\right)$$

Hint : *Click to show hint* : $\cos\left(\frac{n\pi}{2}\right)$ =0 for n odd & ± 1 for even values of n.

Answer: *Click to show answer*: $X(z) = \frac{1}{1 + a^2 z^{-2}} \qquad |z| > |a|$

3) Determine the z-transform of the signal

$$x[n] = -u[-\underline{n}-1] + 0.5n\, u[n]$$

Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane

Solution: *Click to show answer*:

$$X(z) = \sum_{n=0}^{\infty}\left(\frac{0.5}{z}\right)^n - \sum_{n=-\infty}^{-1} z^{-n}$$

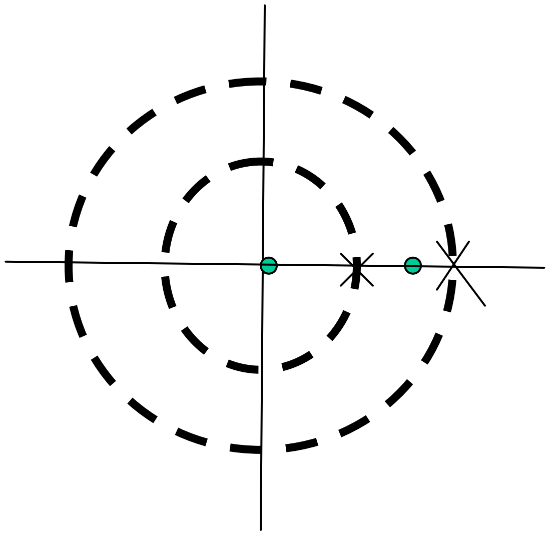
$$= \sum_{n=0}^{\infty}\left(\frac{0.5}{z}\right)^n + 1 - \sum_{k=0}^{\infty} z^k$$

the sum converges, provided that $|z| > 0.5$ and $|z| < 1$.

$$X(z) = \frac{1}{1-0.5z^{-1}} + 1 - \frac{1}{1-z}, \qquad 0.5 < |z| < 1$$

$$= \frac{z(2z-1.5)}{(z-0.5)(z-1)}, \qquad 0.5 < |z| < 1$$

Poles at z=0.5, 1, zeros at z=0, 0.75. ROC is the region in between



Prepared by Dr. Ali Muqaibel