Applications of the Laplace Transform

Transfer Function and Components of System Response

Lecture #36

The material to be covered in this lecture is as follows:

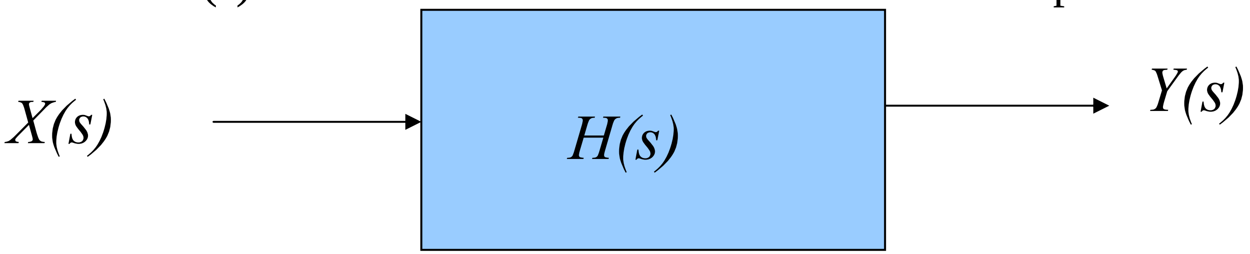
- Concept of transfer function
- Components of system response
- Application of initial value and final value theorems

After finishing this lecture you should be able to:

- Perform circuit analysis to find the transfer function
- Identify the different elements of the system response:
  - Zero-state response vs. zero-input response
  - Transient (natural) vs. steady-state response
- Find the initial value and/or final value of a signal in the Laplace domain.

Transfer Function (T.F)

- By definition, the transfer function of linear time-invariant (LTI) system is the ratio of the Laplace transform of the system output to the Laplace transform of the system input when all initial conditions are zero.
- Let $X(s)$ be the Laplace transform of the input, $Y(s)$ be the Laplace transform of the output, and $H(s)$ be the transfer function. Their relationship is shown as:



$$Y(s) = H(s)X(s)$$

$$H(s) = \left.\frac{Y(s)}{X(s)}\right|_{all\ initial\ condition\ zero}$$

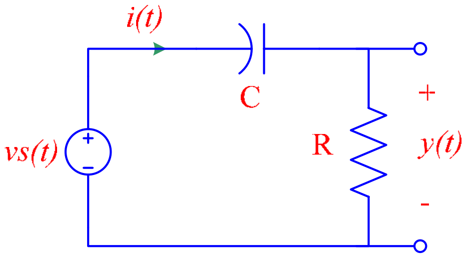
- If $x(t)=\delta(t)$,then $H(s)$ is the Laplace transform of the unit impulse response of the system since $X(s)=1$.

Properties of Transfer Functions

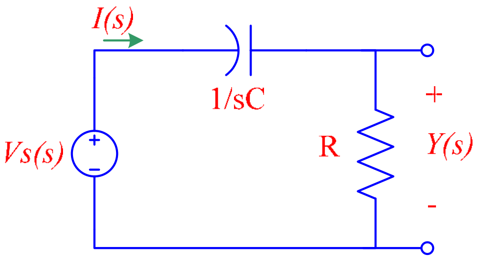
- Provides a complete behavior of a specified linear time invariant system (LTI) system.
- Three important features of the T.F.
  - T.F. is independent of the particular input and is a property of the circuit only.
  - T.F. is obtained for the case of zero initial conditions.
  - T.F. for LTI circuit is a rational function of s.
- The frequency response can be found by letting $s = j\omega$ in $H(s)$.

Example 36.1:

For the system represented by the RC circuit shown in the Figure, find the transfer function. Consider $v_s(t)$ to be the input and $y(t)$ to be the output signals



Represent the system in Laplace domain with zero initial conditions *(animate if possible)*



By voltage divider

$$Y(s) = \frac{R}{R + 1/sC} Vs(s) = \frac{s}{s + 1/RC} Vs(s)$$

Rearranging, the transfer function is given by

$$H(s) = \frac{Y(s)}{Vs(s)} = \frac{s}{s + 1/RC}$$

Component of response

The Laplace transform of differential equation with initial condition is given by:

$$a_n[s^nY(s)-s^{n-1}y(0^-)-s^{n-2}\dot{y}(0^-)-\cdots-y^{(n-1)}(0^-)]$$
$$+a_{n-1}[s^{n-1}Y(s)-s^{n-2}y(0^-)-s3\dot{y}(0^-)-\cdots-y^{(n-2)}(0^-)]$$
$$+....$$
$$+a_1[sY(s)-y(0^-)]+a_0Y(s)$$
$$=[b_ms^m+b_{m-1}s^{m-1}+.....+b_0]X(s)$$

This can be written in the following form

$$D(s)Y(s)-C(s)=N(s)X(s)$$

$C(s)$ terms come from the initial conditions

Solving for the output, we may write

$$Y(s)=\frac{C(s)}{D(s)}+\frac{N(s)}{D(s)}X(s)$$

$$Y(s)=\frac{C(s)}{D(s)}+H(s)X(s)$$

Continue ….Component of response

From the last equation, reproduced here

$$Y(s) = \frac{C(s)}{D(s)} + H(s)X(s)$$

we can distinguish two main components of the system response

The first term (animate) is due to the initial conditions. This part represents the total response if no input is applied. This is why we call this part of the response the ***zero-input response,*** (ZIR). In time domain

$$y_{zero\ input\ response}(t) = L^{-1}\left[\frac{C(s)}{D(s)}\right]$$

The second term (animate) is due to the input. It represents the total response under zero-initial conditions. Hence the name ***zero- state response*** (ZSR). In time domain

$$y_{zero\ state\ response}(t) = L^{-1}\left[H(s)X(s)\right]$$

In summary, total response equals to the zero input response in addition to the zero-state response ,

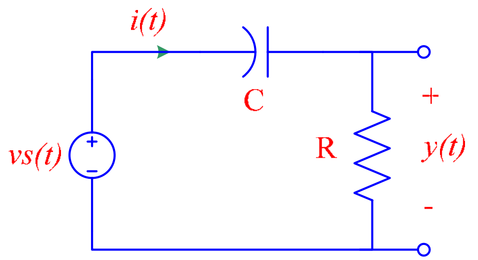
$$y(t) = y_{zero\ input\ response}(t) + y_{zero\ state\ response}(t)$$

Example 36.2:

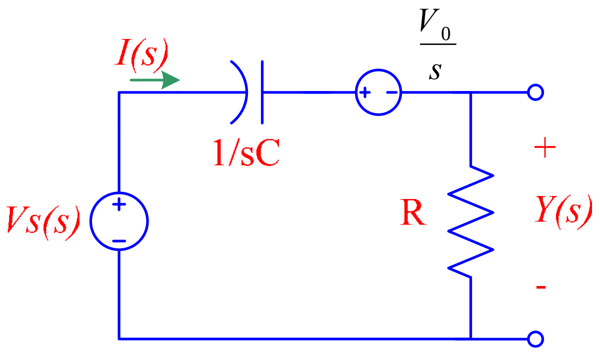
For the same system used in Example 36.1., find the output voltage. Given $v_s(t) = 5\cos 2t\, u(t)$, and an initial capacitor voltage of v_0 with RC=1sec. Express the output as the sum of the ZIR and ZSR.

Solution

For convenience the system in Example 36.1 represented by the RC circuit is reproduced in the Figure below



Equivalent circuit in Laplace domain (animate if possible)



Continue ..Example 36.2:

By KVL , (animate)

$$-V_s(s)+I(s)\frac{1}{sC}+\frac{V_0}{s}+I(s)R=0$$

Solving for the current

$$I(s)=\frac{V_s(s)-\frac{V_0}{s}}{\frac{1}{sC}+R}$$

By Ohm's law

$$Y(s)=I(s)R$$

Substituting the expression for the current (animate) in the last equation (animate)

$$Y(s)=R\left[\frac{V_s(s)-\frac{V_0}{s}}{\frac{1}{sC}+R}\right]$$

Rearranging

$$Y(s)=\frac{s}{s+\frac{1}{RC}}V_s(s)-\frac{V_0}{s}\frac{s}{s+\frac{1}{RC}}$$

Recall from Example 36.1 that $H(s)=\frac{s}{s+1/RC}$ and can rewrite the last equation as

$$Y(s) = H(s)\left[V_s(s) - \frac{V_0}{s}\right]$$

Next step is to find the output for the specific given input

Laplace transform of $v_s(t) = 5\cos 2t\, u(t)$ is $\frac{5s}{s^2+4}$, remember that RC is given to be 1

The previous equation $Y(s) = \frac{s}{s+\frac{1}{RC}} V_s(s) - \frac{V_0}{s}\frac{s}{s+\frac{1}{RC}}$ becomes

$$Y(s) = \frac{5s^2}{(s+1)(s^2+4)} - \frac{V_0}{(s+1)}$$

Using partial fraction expansion

$$Y(s) = \frac{1}{(s+1)} + 4\left[\frac{s}{(s^2+4)} - \frac{1}{2}\frac{2}{(s^2+4)}\right] - \frac{V_0}{(s+1)}$$

By inverse Laplace transform, the output in time domain is

$$y(t) = \left\{e^{-t} + 4\left[\cos 2t - \frac{1}{2}\sin 2t\right]\right\}u(t) - V_0 e^{-t}u(t)$$

The 1st term (animate) is due to the input: ZSR: $y_{ZSR} = \left\{e^{-t} + 4\left[\cos 2t - \frac{1}{2}\sin 2t\right]\right\}u(t)$

The 2nd term (animate)is due to the initial condition: ZIR: $y_{ZIR} = -V_0 e^{-t}u(t)$

Componets of systems response: Transient and Steady State Responses

Note: the output of the previous example may be rewritten as

$$y(t) = \left\{ (1 - v_0)e^{-t} + 4\left[\cos 2t - \frac{1}{2}\sin 2t \right] \right\} u(t)$$

We can now identify the transient and steady-state response as corresponding to the exponential term (animate) and sinusoidal terms respectively. The former goes away as $t \to \infty$.

Unlike the steady-state response, the form of the transient response does not depend on the forcing function (source), but instead is dependent upon the circuit's configurations and elements values.

The transient response is also known as the natural response.

The steady-state response is known as the forced response because it depends on the source

The components of the system response may be written as

Total response=ZSR+ZIR

Total response=transient (natural) response+ steady-state (forced response)

Initial and Final Values

- **Values of $f(t)$ as $t \to 0$ and $t \to \infty$ may be computed from its Laplace transform $F(s)$**
- **Initial value theorem**
  If $f(t)$ and its derivative df/dt have Laplace transforms, then
  $$f\left(0^+\right) = \lim_{s\to\infty} sF\left(s\right)$$
  provided that the limit on the right-hand side of the equation exists.
- **Final value theorem**
  If both $f(t)$ and df/dt have Laplace transforms, then
  $$\lim_{t\to\infty} f\left(t\right) = \lim_{s\to 0} sF\left(s\right)$$
  provided that $s\,F(s)$ has no poles in the right-half of the plane or on the imaginary axis.
- **Recall**
  - Poles are located at values of s which make the denominator equal to zero.
  - Zeros are located at values of s which make the nominator equal to zero.

Example 36.3

Given the signal represented by its Laplace domain transform

$$Y(s)=\frac{10(2s+3)}{s(s^2+2s+5)}$$

Determine the initial and the final values.

For the given Laplace expression, there are:

Poles at s = 0, s = -1 ± j2

Zero at s = -3/2

Initial Value $f(0^+)=\lim_{s\to\infty} sF(s)$

$$y(0^+)=\lim_{s\to\infty}[sY(s)]$$

$$=\lim_{s\to\infty}\left[\frac{10(2s+3)}{s^2+2s+5}\right]$$

$$=\lim_{s\to\infty}\left[10\frac{\left(\frac{2}{s}+\frac{3}{s^2}\right)}{1+\frac{2}{s}+\frac{5}{s^2}}\right]$$

$$=0$$

The limit on the right-hand side of the equation exists

Final Value $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Because $sY(s)$ has no poles in the right-half of the plane or on the imaginary axis, the theorem is applicable.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[sY(s)\right]$$

$$= \lim_{s\to 0}\left[\frac{10(2s+3)}{s^2+2s+5}\right]$$

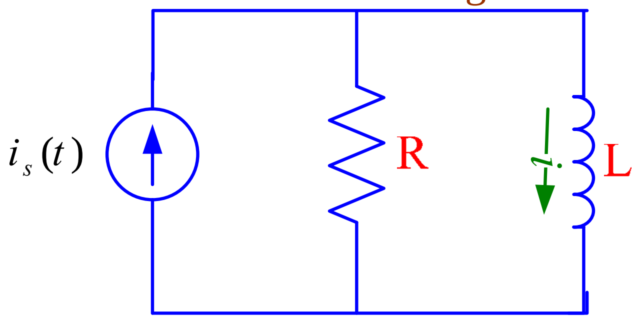
$$= \frac{30}{5}$$

$$= 6$$

Initial value =0 and final value=6.

Self Test:

For the system represented by the *RL* circuit shown in the Figure:



1) Find the transfer function. Consider $i_s(t)$ to be the input and $i(t)$ to be the output signals.
2) For a unit step input, find the zero-state response $i(t)$.
3) If the initial current across the inductor is $i(0)$, find the zero-input response.
4) Find the total response.

Answers (animate)

1) $H(s) = \dfrac{I(s)}{I_s(s)} = \dfrac{R}{R + Ls} = \dfrac{R/L}{R/L + s}$

2) $i_{ZSR}(t) = \left(1 - e^{-Rt/L}\right)u(t)$

3) $i_{ZIR}(t) = i(0)e^{-Rt/L}u(t)$

4) $i_{total}(t) = i_{ZSR}(t) + i_{ZIR}(t) = \left(1 - e^{-Rt/L}\right)u(t) + i(0)e^{-Rt/L}u(t)$