Applications of the Laplace Transform

Analysis of Electrical Networks with Initial Conditions

Lecture #35

The material to be covered in this lecture is as follows:

- Network analysis using Laplace Transform with initial-conditions.
- More examples on circuit analysis including dependent sources and/or op-amps.
- Finding the response of the circuit due to different inputs in the Laplace domain.

After finishing this lecture you should be able to:

- Draw the equivalent Laplace transform for electrical networks with sources and initial conditions.
- Apply electrical network concepts; Thevinin equivalent, Norton equivalent, source transformation, …. etc. in the Laplace domain.
- Perform circuit analysis of problems involving both initial conditions and sources using the concept of Laplace transformed network.

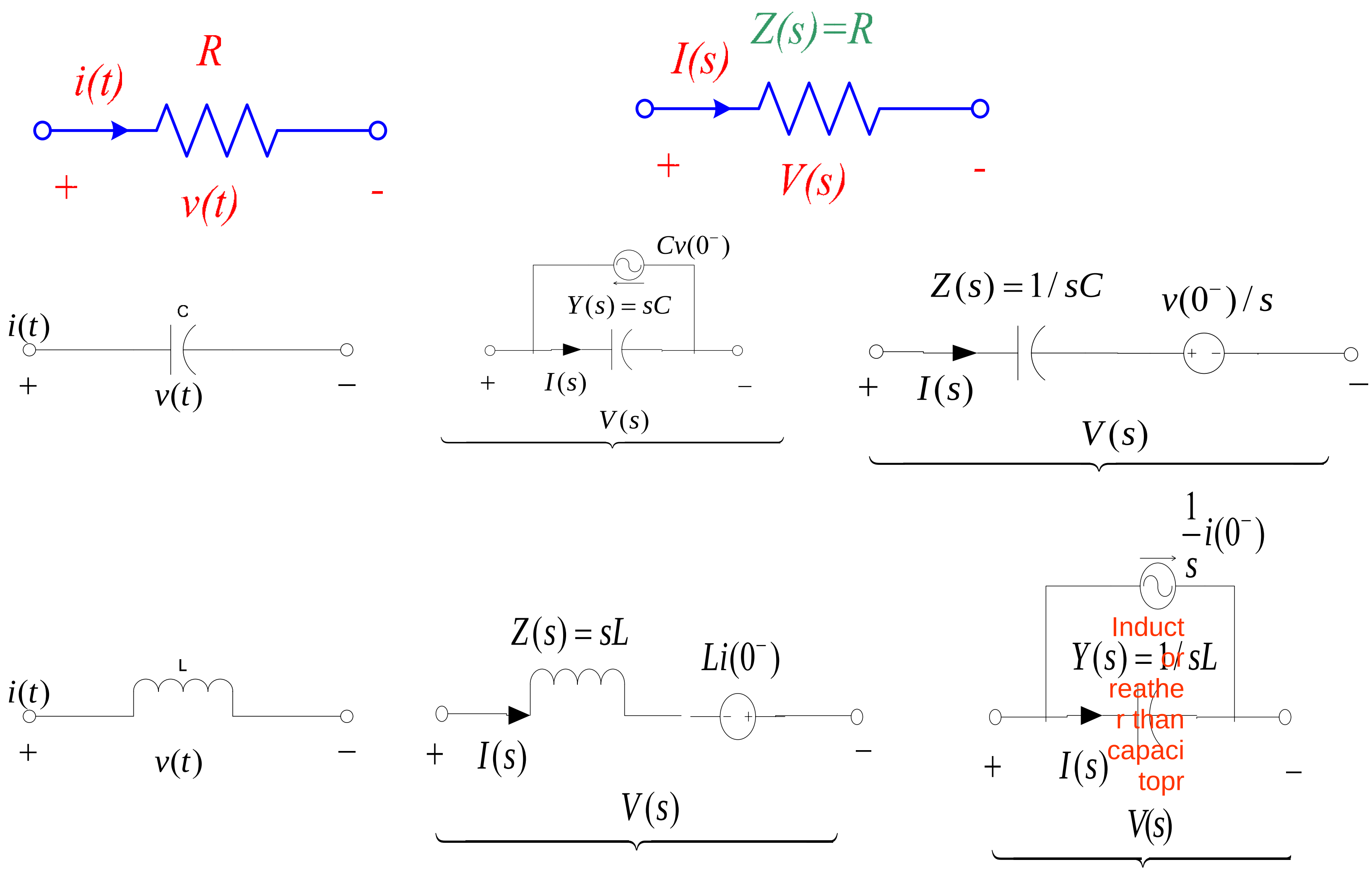
Network Analysis using the Laplace Transform

- In this lecture we consider the more general form of circuit analysis in the Laplace domain.
- Problems involving both initial conditions and sources are considered
- Remember, the first step is to redraw the circuit in the Laplace domain then Ohm's law can be applied to the resulting impedances.
- For the capacitor or inductor we should choose the model that will make the analysis simpler whenever possible.\

- Do you still remember the equivalent Laplace transform for the capacitor, inductor, and resistor?

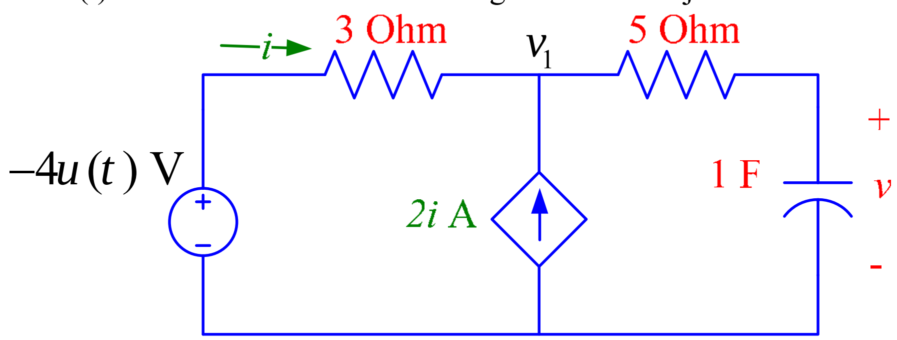
(Click next for a summary of the models with initial conditions. This the same slide form Lecture 34)

– Circuit Elements Models

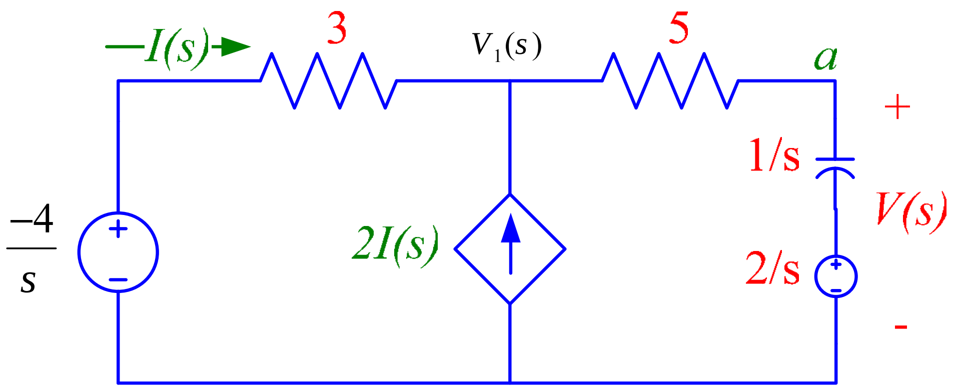


Example 35.1: Circuit with Dependent Source

Find $v(t)$ for the circuit shown in the figure below subject to the initial conditions $v(0)$=2 V.



In the frequency domain, we may redraw the circuit as shown.



- **Note** the voltage to be determined is the voltage across the series combination of a 1 F capacitor and a voltage source having a value of v(0)/s=2/s.

Continue …Solution for Example 35.1

By KCL at node $V_1(s)$ *(animate)*

$$I(s)+2I(s)=\frac{V_1(s)-V(s)}{5}$$

from which

$$3I(s)=3\left[\frac{-4/s-V_1(s)}{3}\right]=\frac{V_1(s)-V(s)}{5}$$

simplifying this expression, we get

$$6V_1(s)-V(s)=\frac{-20}{s} \qquad (35.1)$$

again by KCL at node *a*, *(animate)*

$$\frac{V_1(s)-V(s)}{5}=\frac{V(s)-2/s}{1/s}$$

from which

$$V_1(s)-(1+5s)V(s)=-10 \qquad (35.2)$$

By combining Equation (35.1) and Equation (35.2) we get

$$V(s)-6(1+5s)V(s)=\frac{20}{s}-60$$

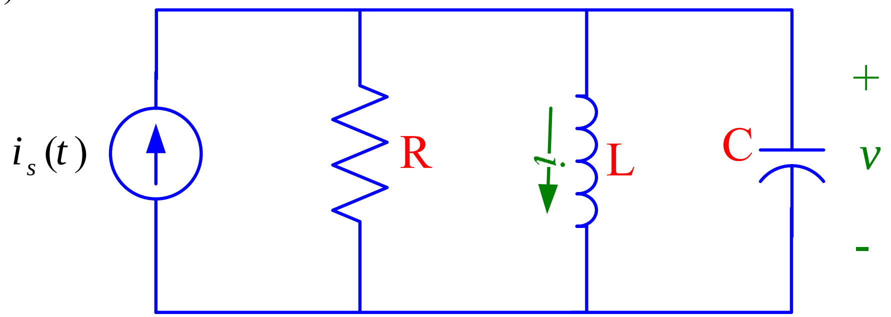
$$\Rightarrow V(s)=\frac{2(s-1/3)}{s(s+1/6)}=\frac{-4}{s}+\frac{6}{s+1/6}$$

By inverse Fourier transform,

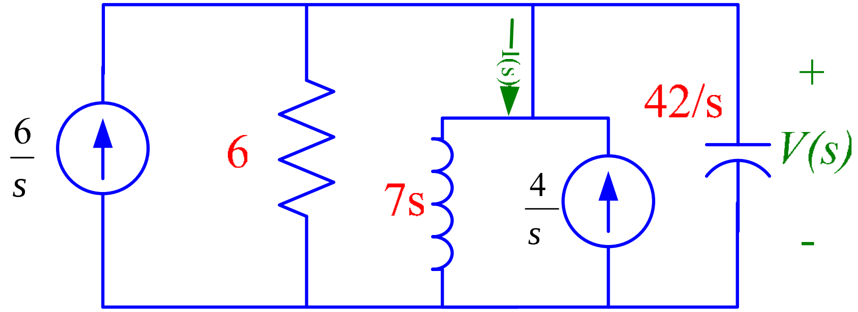
$$v(t)=-4u(t)+6e^{-t/6}u(t)=(-4+6e^{-t/6})u(t)\ \text{V}$$

Example 35.2: Parallel *RLC* Circuit

For the parallel *RLC* shown in the figure, suppose that $i(0)=-4$ A and $v(0)=0$ V. Find $v(t)$ and $i(t)$ for the case that $R=6\Omega$, $L=7$ H, $C=1/42$ F, and $is(t)=6\,u(t)$ A.



The Laplace transformed circuit is shown in the Figure . *(animate)*



- Note the parallel model is selected because it made the analysis easier since the circuit has all parallel elements.
- Note the direction of the parallel source in accordance with the model.
- Reversing the direction of the current source is equivalent to multiplying its value by a minus sign.

By KCL, *(animate)*

$$\frac{6}{s} = \frac{V(s)}{6} + \frac{V(s)}{7s} - \frac{4}{s} + \frac{V(s)}{42/s}$$

Rearrange

$$\frac{6}{s} - \frac{4}{s} = \left(\frac{1}{6} + \frac{1}{7s} + \frac{s}{42}\right)V(s)$$

Combine fractions and simplify

$$\frac{10}{s} = \frac{7s + 6 + s^2}{42s}V(s)$$

Now solving for the voltage,

$$V(s) = \frac{420}{s^2 + 7s + 6}$$

Using partial fraction expansion we may write

$$V(s) = \frac{420}{(s+1)(s+6)} = \frac{K_1}{s+1} + \frac{K_2}{s+6}$$

$$k_1 = \left.\frac{420}{s+6}\right|_{s=-1} = \frac{420}{5} = 84$$

$$k_2 = \left.\frac{420}{s+1}\right|_{s=-6} = \frac{420}{-5} = -84$$

Using inverse Laplase transform,

$$v(t) = 84e^{-t}u(t) - 84e^{-6t}u(t) = 84(e^{-t} - e^{-6t})u(t) \text{ V}$$

To find the current, remember to consider the current f the two branches in the model

$$I(s)=\frac{V(s)}{7s}-\frac{4}{s}=\frac{1}{7s}\left(\frac{420}{(s+1)(s+6)}\right)-\frac{4}{s}=\frac{60}{s(s+1)(s+6)}-\frac{4}{s}$$

$$=\frac{60-4(s+1)(s+6)}{s(s+1)(s+6)}=\frac{-4s^2-28s+36}{s(s+1)(s+6)}=\frac{k_0}{s}+\frac{k_1}{s+1}+\frac{k_2}{s+6}$$

The constants for the partial fraction expansion are found as follows:

$$k_0=\left.\frac{-4s^2-28s+36}{(s+1)(s+6)}\right|_{s=0}=\frac{36}{6}=6$$

$$k_1=\left.\frac{-4s^2-28s+36}{s(s+6)}\right|_{s=-1}=\frac{-4+28+36}{(-1)(5)}=-12$$

$$k_2=\left.\frac{-4s^2-28s+36}{s(s+1)}\right|_{s=-6}=\frac{-144+168+36}{-6(5)}=2$$

By inverse Laplace transform

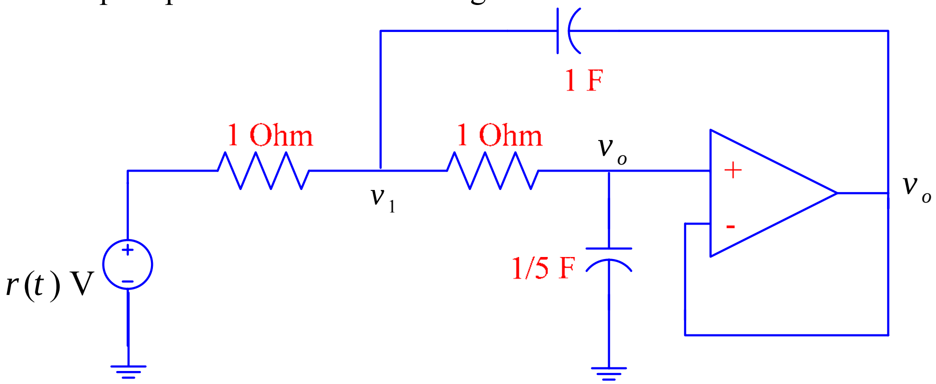
$i(t)=6u(t)-12e^{-t}u(t)+2e^{-6t}u(t)=2(3-6e^{-t}+e^{-6t})u(t)$ A

As a practice you might verify the relation between the voltage and the current of the inductor. Remember

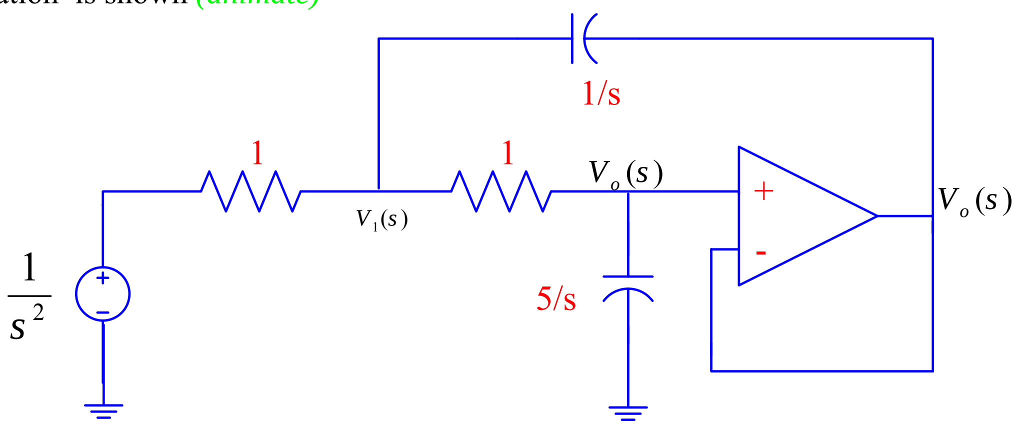
$$v_L(t)=L\frac{di_L(t)}{dt}$$

Example 35.3: Circuit with Op-amp

Determine the ramp response *vo*(*t*) for the op-amp circuit shown in the figure below. Assume zero-initial conditions



Ans.
The frequency domain representation is shown *(animate)*



Since $r(t) = t\,u(t)$, then the Laplace transform is $1/s^2$.

By KCL at node $V_1(s)$, (animate if possible)

$$\frac{V_1(s)-1/s^2}{1}+\frac{V_1(s)-V_o(s)}{1}+\frac{V_1(s)-V_o(s)}{1/s}=0$$

From which

$$(s+2)V_1(s)-(s+1)V_o(s)=\frac{1}{s^2} \quad (35.3)$$

The voltage across the capacitor labeled 5/s is

$$V_o(s)=\frac{5/s}{1+5/s}V_1(s)=\frac{5}{s+5}V_1(s)\Rightarrow V_1(s)=\frac{s+5}{5}V_o(s)$$

Now substituting the expression for $V_1(s)$ into Equation (35.3) we get

$$(s+2)\frac{s+5}{5}V_o(s)-(s+1)V_o(s)=\frac{1}{s^2}$$

Solving for the output voltage

$$Vo(s)=\frac{5}{s^2(s^2+2s+5)}$$

Using partial fraction expansion you should be able to show that

$$V_o(s)=\frac{-2/5}{s}+\frac{1}{s^2}+\frac{2/5(s+1)}{(s+1)^2+2^2}+\frac{1/5(2)}{(s+1)^2+2^2}$$

Hence the ramp response can be found using inverse Laplace transform

$$vo(t)=-\frac{2}{5}u(t)+tu(t)+\frac{2}{5}e^{-t}\cos 2t\,u(t)+\frac{1}{5}e^{-t}\sin 2t\,u(t)$$

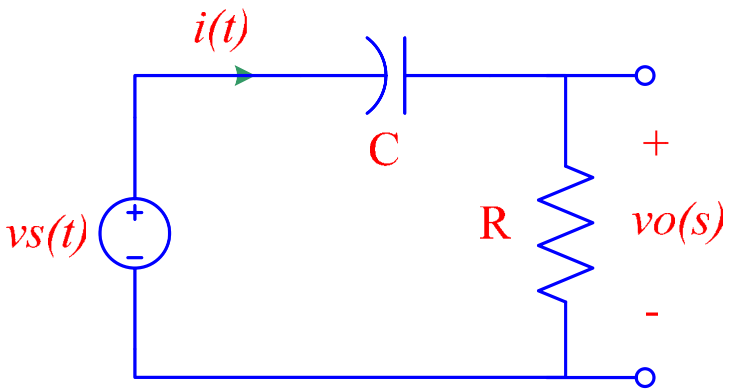
$$=\left[-\frac{2}{5}+t+e^{-t}\left(\frac{2}{5}\cos 2t+\frac{1}{5}\sin 2t\right)\right]u(t)\ \text{V}$$

Practice

Consider the RC circuit shown in the Figure. Instead of the output is taken across the resistor. Find the output for the input

$v_s(t) = 5\cos 2t\; u(t)\,\text{V}$

And an initial capacitor voltage of v_0=3 V with RC=1 s.



Ans.

Click here to see the answer

$$v_o(t) = \left\{-2e^{-t} + 4\left[\cos 2t - \frac{1}{2}\sin 2t\right]\right\}u(t)$$

Note in the next lecture this problem will be solved in details