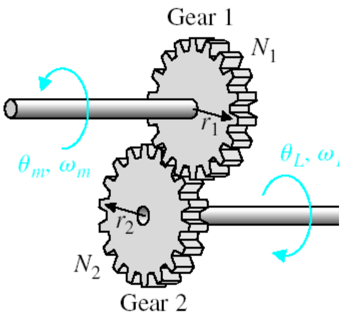
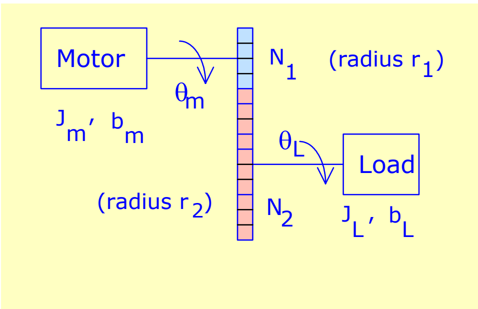
2. Mathematical Models of Systems (cont.)

Transfer Functions for Systems with Gears

A gear reduction is usually required between the high-speed, low-torque servomotor and the load to obtain speed reduction and torque magnification



Consider the figure below, which shows a motor with inertia J_m and damping b_m driving a load consisting of inertia J_L and damping b_L. To obtain an equivalent inertia J, and equivalent damping b, we will reflect the load inertia J_L and the load damping b_L, to the motor shaft.



- As the gears turn, the distance traveled along each gear's circumference is the same. Thus $r_1\theta_m = r_2\theta_L$
  •The number of teeth on the surface of the gears is proportional to the radii. Thus $r_1N_2 = r_2N_1$
- The work done by one gear is equal to that of the other (on the assumption of no losses). Thus $T_m\theta_m = T_L\theta_L$

The above equations lead to

$$\frac{T_m}{T_L} = \frac{\theta_L}{\theta_m} = \frac{r_1}{r_2} = \frac{N_1}{N_2} = n\,; \quad (n \text{ is the gear ratio; } \leq 1)$$

$$T_m = \frac{N_1}{N_2}T_L = \frac{N_1}{N_2}\left(J_L\frac{d^2\theta_L(t)}{dt^2} + b_L\frac{d\theta_L(t)}{dt}\right) = \left[\frac{N_1}{N_2}\right]^2\left(J_L\frac{d^2\theta_m(t)}{dt^2} + b_L\frac{d\theta_m(t)}{dt}\right)$$

$$T_m = \left[\frac{N_1}{N_2}\right]^2 J_L\frac{d^2\theta_m(t)}{dt^2} + \left[\frac{N_1}{N_2}\right]^2 b_L\frac{d\theta_m(t)}{dt}$$

$$T_m = n^2J_L\left(\frac{d^2\theta_m(t)}{dt^2}\right) + n^2b_L\left(\frac{d\theta_m(t)}{dt}\right)$$

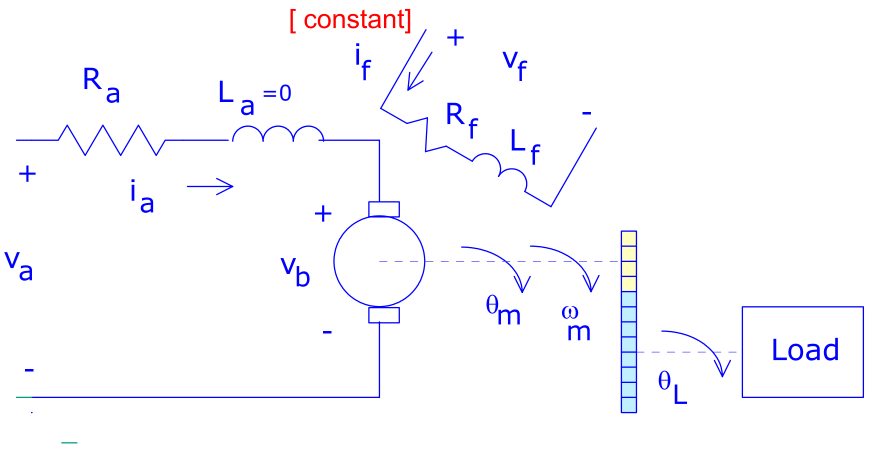
Conclusion :the load inertia $J_L$and the load damping b_L,can be reflected to the to the motor shaft by multiplying them by $\left[\frac{N_1}{N_2}\right]^2 = n^2$.

For the above configuration, The equivalent inertia and damping referred to the motor side are:

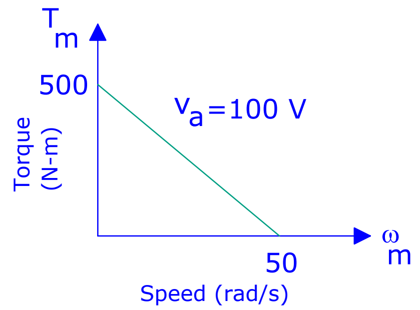
$$J = J_m + \left[\frac{N_1}{N_2}\right]^2 J_L\ ;\ b = b_m + \left[\frac{N_1}{N_2}\right]^2 b_L$$

Example

Given the system and torque-speed curve shown, find the transfer function, $\frac{\theta_L(s)}{V_a(s)}$.



$J_m = 5$ kg-m^2; $J_L = 700$ kg-m^2 ; $b_m = 2$ N-m s/rad ; $b_L = 800$ N-m s/rad ; $n = \frac{100}{1000}$



Solution

The required transfer function is

$$\frac{\theta_L(s)}{V_a(s)} = \frac{nK_m}{s[R_a(Js+b)+K_bK_m]} = \frac{n\frac{K_m}{R_a}}{s[\ (Js+b)+K_b\frac{K_m}{R_a}]}$$

$$J = J_m + n^2 J_L = 5 + 700\left[\frac{100}{1000}\right]^2 = 12 \text{ kg-m}^2$$
$$b = b_m + n^2 b_L = 2 + 800\left[\frac{100}{1000}\right]^2 = 10 \text{ N-m s/rad}$$

To find the electrical constant $\frac{K_m}{R_a}$ and K_b, we make use of the torque-speed curve.

Remember that with $L_a = 0$, $T_m = \frac{K_m}{R_a}(v_a - K_b\omega_m)$

When $\omega_m = 0, T_m = \frac{K_m}{R_a}(v_a) \Rightarrow$ stall torque

Hence $\frac{K_m}{R_a} = \frac{T_{stall}}{v_a} = \frac{500}{100} = 5$ N-m /V

When $T_m = 0, v_a = K_b \omega_m$, hence $K_b = \frac{100}{50} = 2$ V s/rad

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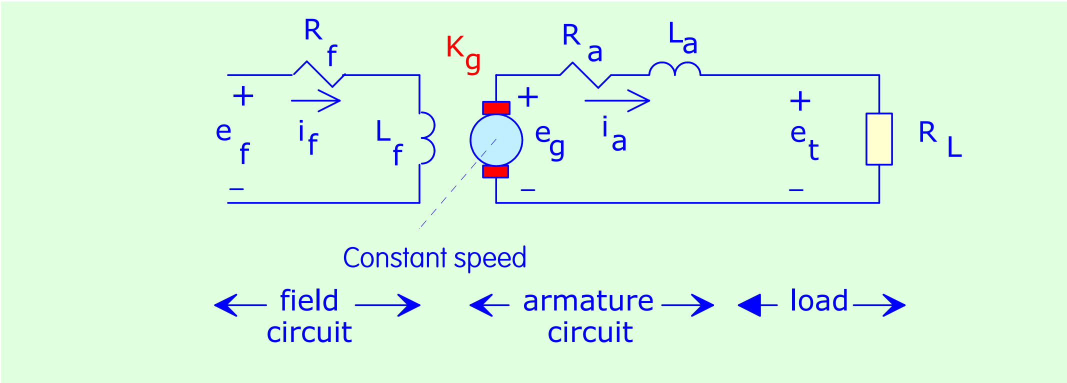
dc generator



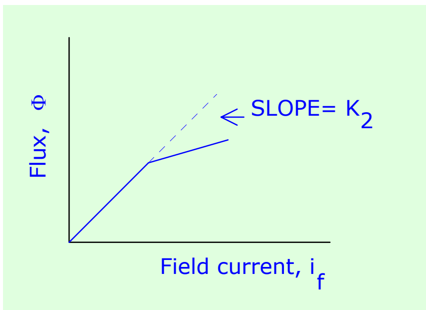
A dc generator can be used as a power amplifier, in which the power required to excite the field is lower than the power required output rating of the armature circuit.
A dc generator is represented schematically in the figure, in which R_f, L_f, and R_g, L_g are the resistance and inductance of the field and armature circuits, respectively.

- The voltage e_g is directly proportional to the product of the magnetic flux ϕ set up by the field and the speed of rotation of the armature. This is expressed by

$$e_g = K_1 \phi \omega$$

- The flux is a function of the field current and the type of iron used in the field. A typical magnetization curve showing flux as a function of field current is shown . Up to saturation the relation is approximately linear, and the flux is directly proportional to field current:

$$\phi = K_2 i_f$$



- When the generator is used as a power amplifier, the armature is driven at a constant speed, and The voltage e_g is directly proportional to the field current

$$e_g = K_g i_f \quad ; \quad E_g(s) = K_g I_f(s)$$

- The equations of the generator are
  $E_f(s) = (R_f + L_f s)I_f(s)$; $E_t(s) = E_g(s) - (R_a + L_a s)I_a(s)$; $E_t(s) = I_{a(s)}R_L$

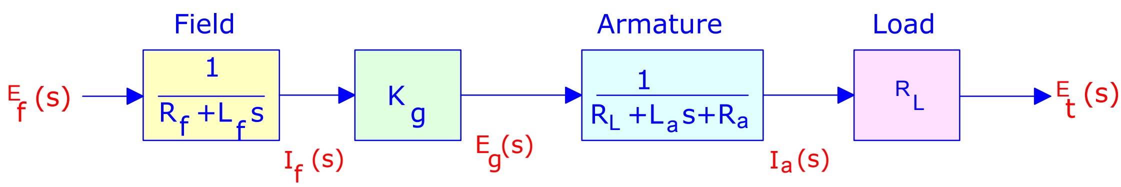
  The transfer functions are

  $$\frac{E_t(s)}{E_f(s)} = \frac{K_g R_L}{(R_f + L_f s)(R_a + L_a s + R_L)} \; ; \left(\frac{E_g(s)}{E_f(s)} = \frac{K_g}{(R_f + L_f s)} \quad \text{applies at no-load} \right)$$

  The power gain is given by

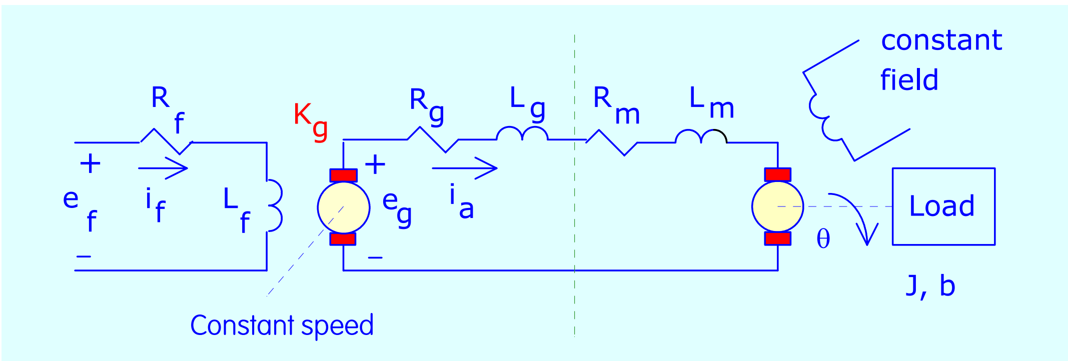
  $$\frac{p_o}{p_i} = \frac{e_t i_a}{e_f i_f}$$

The block-diagram model of the dc generator is shown next



<u>The Ward-Leonard System</u>

A configuration having a dc generator driving an armature-controlled dc motor is known as a Ward-Leonard System. The dc generator acts as a rotating power amplifier that supplies the power which, in turn, drives the servomotor.



- To enable us to combine the transfer function relationships derived previously for the dc generator and armature-controlled dc motor, we assume that the generator voltage $e_{g}(t)$ is applied directly to the armature of the motor. Therefore, we are interested in applying the generator equation $\frac{E_g(s)}{E_f(s)} = \frac{K_g}{(R_f + L_f s)}$.

- In order to apply the motor transfer function derived earlier, we must first combine the resistive and inductive components of the generator's and motor's armatures. This will result in a set of new modified time constants as follows;

$$\frac{\theta(s)}{E_g(s)} = \frac{K_m}{s[(R_t + sL_t)(Js + b) + K_b K_m]}$$

Where

$R_t = R_g + R_m$; and $L_t = L_g + L_m$

It is now relatively simple to obtain the transfer-function representation of configuration as follows:

$$\frac{\theta(s)}{Ef(s)} = \frac{K_m}{s[(R_t + sL_t)(Js + b) + K_b K_m]} \frac{K_g}{(R_f + L_f s)}$$

The block diagram of the Ward-Leonard system is