8. FREQUENCY RESPONSE METHODS

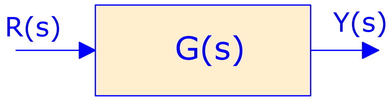
CONCEPT

A very practical and important approach to the analysis and design of a system is the **frequency response** method .

> The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. For a linear system, the resulting output signal, as well as signals throughout the system, are sinusoidals of the same frequency in the steady state; they differ from the input wave form only in amplitude and phase angle.

ANALYTICAL EXPRESSION FOR FREQUENCY RESPONSE

The transfer function of a system $G(s)$ can be described in the frequency domain by the relation



$$G(j\omega) = G(s)|_{s=j\omega}$$

The transfer function $G(j\omega)$ can be represented by its real and imaginary parts

$$G(j\omega) = \mathrm{Re}|G(j\omega)| + j\,\mathrm{Im}|G(j\omega)|$$
$$= R(\omega) + jX(\omega)$$

or alternatively by its magnitude and phase:

$$G(j\omega) = |G(j\omega)|\ e^{j\phi(j\omega)} = |G(\omega)|\ \angle\phi(\omega)$$

Where

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)}\ ,\quad \text{and} \quad |G(j\omega)| = \sqrt{|R(\omega)|^2 + |X(\omega)|^2}$$

PLOTTING FREQUENCY RESPONSE

The frequency response of the system $G(j\omega)$ can be portrayed graphically in two ways:

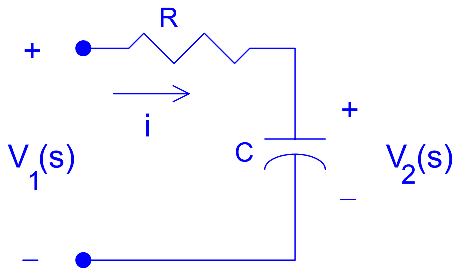
- As a **polar plot**, where the phasor length is the magnitude $|G(\omega)|$, and the phasor angle is the phase $\phi(\omega)$. [or alternatively, the coordinates of the polar plot are the real part of $R(\omega)$, and the imaginary part $X(\omega)$.]

- As **logarithmic plots**, often called **Bode plots**, representing separately the magnitude and phase, as a function of frequency. The magnitude curve can be plotted in decibels (**dB**) vs. log ω, where $dB = 20\log|G(j\omega)|$. The phase curve is plotted as phase angle vs. log ω .

Two examples of polar plots are given next:

Example 1   Frequency response of an RC filter

A simple RC filter us shown. The transfer function of this filter is



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1}$$

The sinusoidal steady-state transfer function is

$$G(j\omega) = \frac{1}{j\omega(RC)+1} = \frac{1}{j(\frac{\omega}{\omega_1})+1} \quad ; \quad \omega_1 = \frac{1}{RC}$$

$$G(j\omega) = R(\omega) + jX(\omega) = \frac{1}{1+\left(\frac{\omega}{\omega_1}\right)^2} - \frac{j(\frac{\omega}{\omega_1})}{1+\left(\frac{\omega}{\omega_1}\right)^2} \quad \text{or}$$

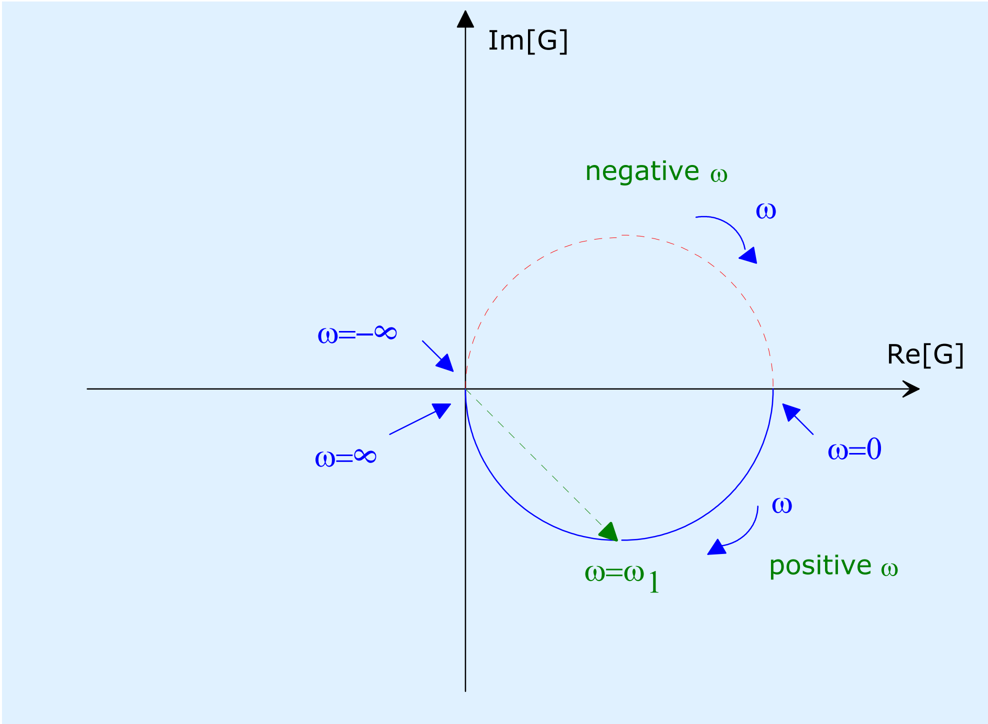
$$G(j\omega) = |G(\omega)| \angle \phi(\omega) = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}} \angle -\tan^{-1}\left(\frac{\omega}{\omega_1}\right)$$

The phase angle and the magnitude are readily calculated at the frequencies $\omega = 0, \omega_1, and\ \infty$. The values are given in the table.

ω	0	ω_1	$+\infty$
$\|G(\omega)\|$	1	$\frac{1}{\sqrt{2}}$	0
$\phi(\omega)$	0°	-45°	-90°

Note that for $0 < \omega < \infty$, $-90^\circ < \phi(\omega) < 0^\circ$. This indicates that the entire polar plot lies in the fourth quadrant.

The polar plot for the RC filter is shown.



Example 2

Consider the transfer function

$$G(s) = \frac{K}{s(\tau s + 1)}$$

The sinusoidal steady-state transfer function is

$$G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{j\omega - \omega^2\tau}$$

Then the magnitude and phase angle are written as

$$|G(\omega)| = \frac{1}{\sqrt{\omega^2 + \omega^4\tau^2}}$$

$$\phi(\omega) = -\tan^{-1}\left(\frac{1}{-\omega\tau}\right)$$

The phase angle and the magnitude are readily calculated at the frequencies $\omega = 0, \omega = \frac{1}{\tau}, and\ \infty$. The values are given in the table.

ω	0	$\frac{1}{\tau}$	$+\infty$
$\lvert G(\omega)\rvert$	∞	$\frac{K\tau}{\sqrt{2}}$	0
$\phi(\omega)$	-90°	-135°	-180°

Note that for $0 < \omega < \infty$, $-90^\circ > \phi(\omega) > -180^\circ$. This indicates that the entire polar plot lies in the third quadrant.

The polar plot for the transfer function is shown.