6. THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Concept of Stability

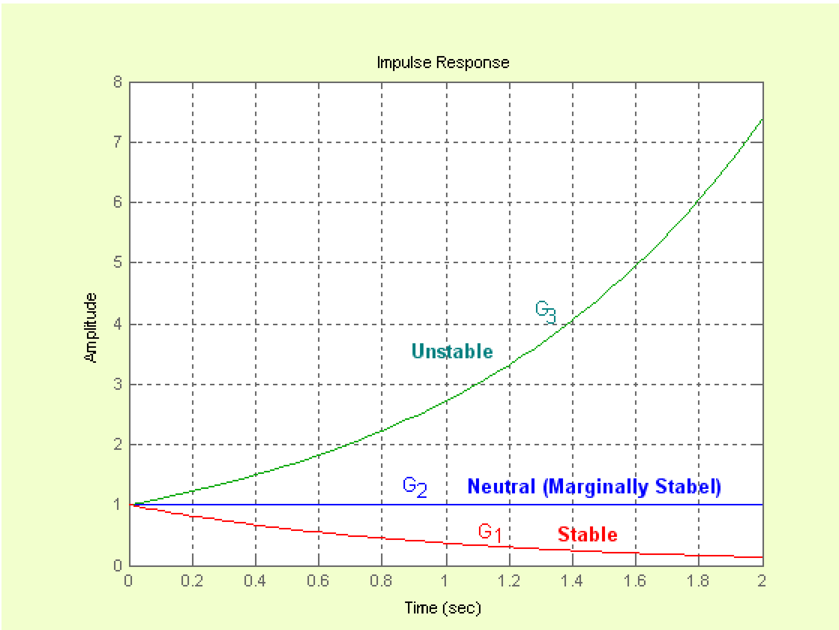
Stability is of extreme importance when considering the design and analysis of feedback control systems.

- A stable system is defined as a system with bounded (limited) response to a bounded (limited) input.

If a dynamic system is subjected to an input, or initial condition, its response may be decreasing, neutral, or increasing. The location in the s-plane of the poles of a system indicates the resulting transient response.

- Poles in left-hand portion of the s-plane result in decreasing response
- Poles on $j\omega$-axis result in neutral response
- Poles in right-hand portion of the s-plane result in increasing response

Consider the systems $G_1(s) = \frac{1}{s+1}$, $G_2(s) = \frac{1}{s}$, and $G_3(s) = \frac{1}{s-1}$. The impulse response of the three systems is shown.



Clearly the poles of desirable dynamic systems must lie in the left-hand portion of the s-plane.

Absolute Stability

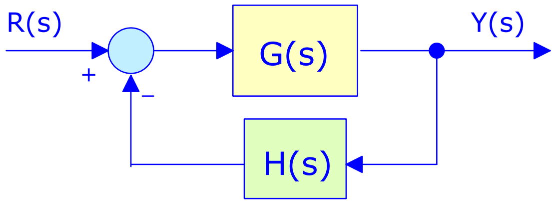
A closed-loop feedback system is either STABLE or UNSTABLE. This type of characterization is referred to as **absolute stability.** (a system that is absolutely stable is called stable system - the label absolute is dropped.

Relative Stability

Given that a closed-loop system is stable, the degree of stability can be further characterize. This is referred to as **relative stability**.

Determination of Stability

To determine the stability of a feedback system, one could determine the roots of characteristic equation:



$q(s) = 1 + G(s)H(s) = 0$

This will answer the question "Is the system stable or not?", but we have determined much more information than is necessary. Therefore several methods exist to provide the required yes or no to the stability question.

Routh-Hurwitz Stability Criterion

The Routh-Hurwitz criterion answers the question "Is the system stable or not?", without actually solving the characteristic equation for the roots. It is based on ordering the coefficients of the characteristic equation

$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 = 0$

into an array as follows:

s^n	a_n	a_{n-2}	a_{n-4}	$\ldots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\ldots$
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	$\ldots$
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	
$\ldots$	$\ldots$	$\ldots$	$\ldots$	
s^0	h_{n-1}			

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \quad ; \quad b_{n-3} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_{n-1} = \frac{b_{n-1}a_{n-3} - a_{n-1}b_{n-3}}{b_{n-1}}$$

and so on ...

A Necessary condition for Stability

- All $\{a_i\}$must be positive
- All $\{a_i\}$must be nonzero

A Necessary and sufficient condition for Stability

The Routh-Hurwitz criterion states that the number of roots of $q(s)$ with positive real parts is equal to the number of changes in sign of the first column of the array. [A pattern of $+,-,+$ is counted as two sign changes]

For a stable system → no changes in sign in the first column

Four distinct cases or configurations of the first column array must be considered, and each must be treated separately and require suitable modifications of the array calculation procedure:

(1) No element in the first column is zero;
(2) there is a zero in the first column, but some other elements of the row containing the zero in the first column are nonzero;
(3) there is a zero in the first column, and the other elements of the row containing the zero are also zero; and
(4) as in (3) with repeated roots on the $j\omega$-axis.

Case 1. No element in the first column is zero

Example

Determine whether any of the roots of the characteristic equation
$s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 = 0$
are in the RHP.

Solution

The Routh Array for this C.E. is:

s^6	1	3	1	4
s^5	4	2	4	0
s^4	5/2	0	4	
s^3	2	-12/5	0	
s^2	3	4		
s	-76/15	0		
s^0	4			

The C.E. has **two** roots in the right-hand portion of the s-plane because there are **two** sign changes.

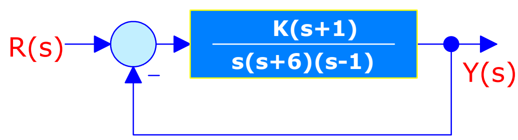
Using MATLAB, one can find the actual roots of the C.E. :

-3.2644
0.6797 + j0.7488
0.6797 - j0.7488
-0.6046 + j0.9935
-0.6046 - j0.9935
-0.8858

This confirm the results of the Routh array

Example

Determine the range of K for which the system is stable.



Solution

The C.E. is

$$1+\frac{K(s+1)}{s(s+6)(s-1)}=0 \Rightarrow s^3+5s^2+(k-6)s+K=0$$

The Routh Array for this C.E. is:

s^3	1	K-6		
s^2	5	K		
s	(4K-30)/5	0	➔	$k>7.5$
s^0	K	➔		$K>0$

For the system to be stable, it is necessary that $K>7.5$