3. STATE VARIABLE MODELS (cont.)

Alternative Signal-Flow Graph Models (cont.)

Diagonal Form

Consider the transfer function:

$$\frac{Y(s)}{R(s)}=\frac{30(s+1)}{s^3+9s^2+26s+24}=\frac{30(s+1)}{(s+5)(s+2)(s+3)}$$

It is clear that the transient response of the system has three modes, These modes are indicated by the partial fraction expansion as

$$\frac{Y(s)}{R(s)}=\frac{k_1}{(s+5)}+\frac{k_2}{(s+2)}+\frac{k_3}{(s+3)}$$

The coefficients $k_1, k_2,$ and k_3 are called residues and are evaluated by multiplying through by the denominator factor of $\frac{30(s+1)}{(s+5)(s+2)(s+3)}$ corresponding to k_i and setting s equal to the root.

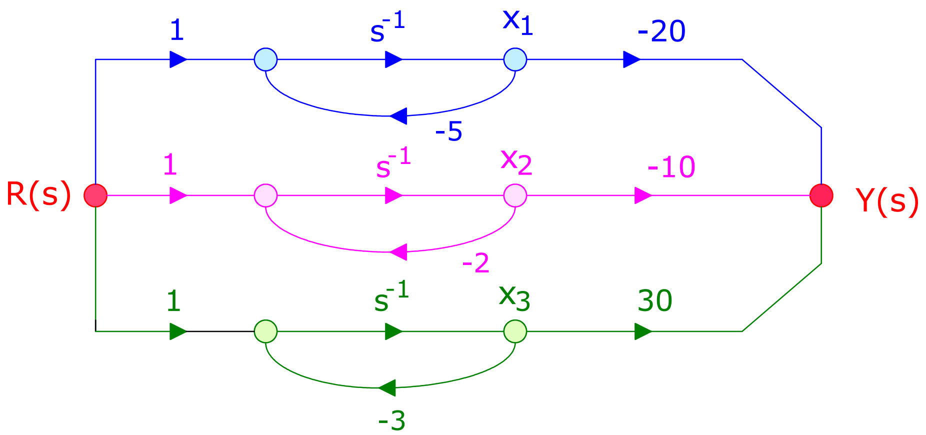
Evaluating $k_1, k_2,$ and k_3 we have

$$k_1=\left[(s+5)\frac{30(s+1)}{(s+5)(s+2)(s+3)}\right]\Bigg|_{s=-5}=-20$$

$$k_2=\left[(s+2)\frac{30(s+1)}{(s+5)(s+2)(s+3)}\right]\Bigg|_{s=-2}=-10$$

$$k_3=\left[(s+3)\frac{30(s+1)}{(s+5)(s+2)(s+3)}\right]\Bigg|_{s=-3}=30$$

$$\frac{Y(s)}{R(s)}=\frac{-20}{(s+5)}+\frac{-10}{(s+2)}+\frac{30}{(s+3)}$$



Using the above SFG to derive the set of first-order differential equations, we obtain:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} -20 & -10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

THE TRANSFER FUNCTION FROM STATE EQUATIONS

Given the state variable equations, we can obtain the transfer function using a signal-flow graph model and applying Mason's rule. We will now derive a formula for the transfer function of a single-input, single-output system.

Given

$\dot{\mathbf{x}} = A\mathbf{x} + Bu$; $y = C\mathbf{x}$ [D is assumed $= 0$]

The Laplace transforms of the above equations are

$$s\mathbf{X}(s) = A\mathbf{X}(s) + BU(s) \; ; \; Y(s) = C\mathbf{X}(s)$$

Note that we do not include initial conditions, since we seek the transfer function.

$$(sI - A)\mathbf{X}(s) = BU(s)$$

$$\mathbf{X}(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C\mathbf{X}(s) = C(sI - A)^{-1}BU(s)$$

$$Y(s) = C\Phi(s)BU(s)$$

Therefore the transfer function is $G(s) = C\Phi(s)B$

If $D \neq 0$, the transfer function is $G(s) = C(sI - A)^{-1}B + D = C\Phi(s)B + D$

Example

Determine the transfer function of the system described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t) \; ; \; y(t) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution

$$[sI-A]=\begin{bmatrix} s & \frac{1}{C} \\ \frac{-1}{L} & \left(s+\frac{R}{L}\right) \end{bmatrix};\ \Delta(s)=|sI-A|=\begin{vmatrix} s & \frac{1}{C} \\ \frac{-1}{L} & \left(s+\frac{R}{L}\right) \end{vmatrix}=s^2+\frac{R}{L}s+\frac{1}{LC}$$

$$\Phi(s)=[sI-A]^{-1}=\begin{bmatrix} s & \frac{1}{C} \\ \frac{-1}{L} & \left(s+\frac{R}{L}\right) \end{bmatrix}^{-1}=\frac{1}{\Delta(s)}\begin{bmatrix} \left(s+\frac{R}{L}\right) & \frac{-1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

Then the transfer function is

$$\frac{Y(s)}{U(s)}=\begin{bmatrix} 0 & R \end{bmatrix}\frac{1}{\Delta(s)}\begin{bmatrix} \left(s+\frac{R}{L}\right) & \frac{-1}{C} \\ \frac{1}{L} & s \end{bmatrix}\begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)}=\frac{\frac{R}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Evaluation of the State Transition Matrix

For higher order systems, evaluating $\Phi(s)$ using the formula $\Phi(s)=[sI-A]^{-1}$ is generally inconvenient. The usefulness of the signal-flow graph state model for obtaining the state transition matrix is highlighted.

Consider the system $\dot{\boldsymbol{x}}=A\boldsymbol{x}+Bu$;
The solution for the above system, when $u(t)=0$, is
$\boldsymbol{x}(t)=\Phi(t)\boldsymbol{x}(o)$

Taking the Laplace transformation of the above equation, we have
$\boldsymbol{X}(s)=\Phi(s)\boldsymbol{x}(0)$

Therefore we can evaluate the Laplace transform of the transition matrix from the signal-flow graph by determining the relation between a state variable $X_i(s)$ and the state initial conditions $[x_1(0), x_2(0) \ldots x_n(0)]$, using Mason's gain formula.

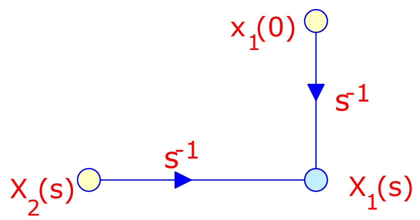
Thus for a second-order system, we would have

$$X_1(s)=\varphi_{11}(s)x_1(0)+\varphi_{12}(s)x_2(0)$$
$$X_2(s)=\varphi_{21}(s)x_1(0)+\varphi_{22}(s)x_2(0)$$

Note that all the elements of the state transition Matrix $\varphi_{ij}(s)$, can be obtained by evaluating the individual relationships between $X_i(s)$ and $x_j(0)$ from the state model flow graph.

How to show Initial Conditions on the SFG

Consider the equation $\dot{x}_1 = x2$; $x_1(0)$ *Taking Laplace transform yields* $sX_1(s) - x_1(0) = X_2(s)$ *The above equation becomes* $X_1(s) = s^{-1}x_1(0) + s^{-1}X_2(s)$, *which is algebraic and can be represented by a signal flow graph as shown.*



Note that the initial condition of the state x_1 *appears as an input to the node representing the state with a branch gain of* s^{-1}.

Example

Determine $\Phi(s)$ for the system given by $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = [0\ 3]$ using two different methods.

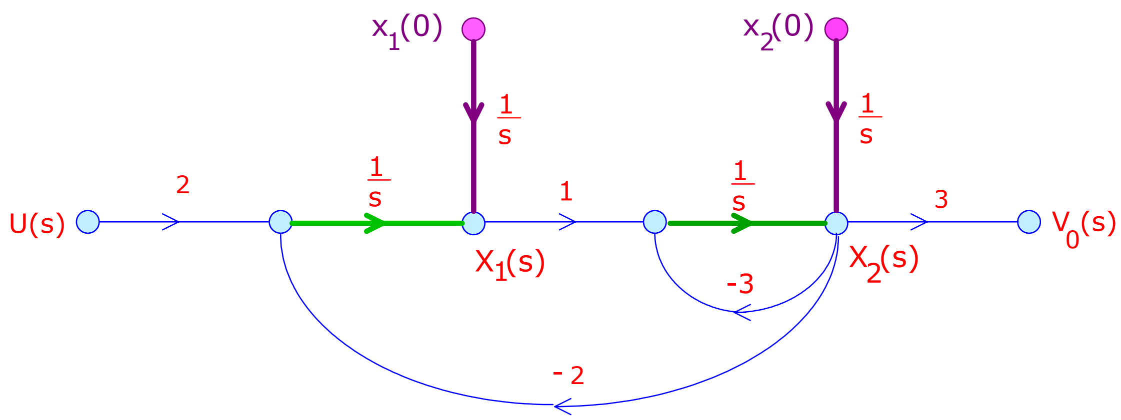
Solution

(1)

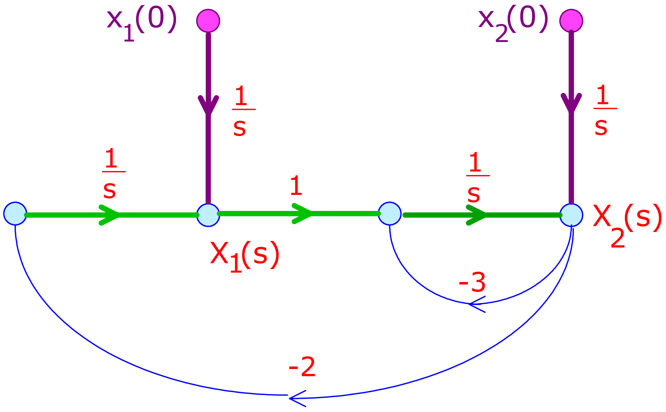
$$[sI - A] = \begin{bmatrix} s & 2 \\ -1 & (s+3) \end{bmatrix}; \Delta(s) = |sI - A| = \begin{vmatrix} s & 2 \\ -1 & (s+3) \end{vmatrix} = s^2 + 3s + 2$$

$$\Phi(s) = [sI - A]^{-1} = \begin{bmatrix} s & 2 \\ -1 & (s+3) \end{bmatrix}^{-1} = \frac{1}{(s^2+3s+2)} \begin{bmatrix} (s+3) & -2 \\ 1 & s \end{bmatrix}$$

(2) Draw a signal-flow graph showing all initial conditions



To obtain $\Phi(s)$, set $U(s) = 0$, and redraw the SFG without the input and output nodes because they are not involved in the evaluation of $\Phi(s)$.



Recall that

$$X_1(s) = \varphi_{11}(s)x_1(0) + \varphi_{12}(s)x_2(0)$$
$$X_2(s) = \varphi_{21}(s)x_1(0) + \varphi_{22}(s)x_2(0)$$

Where $\Phi(s) = \begin{bmatrix} \varphi_{11}(s) & \varphi_{12}(s) \\ \varphi_{21}(s) & \varphi_{22}(s) \end{bmatrix}$

Using Mason's gain formula, we obtain

$$\varphi_{11}(s) = \left.\frac{X_1(s)}{x_1(0)}\right|_{x_2(0)=0} = \frac{\frac{1}{s}(1+3s^{-1})}{1+3s^{-1}+2s^{-2}} = \frac{s+3}{(s^2+3s+2)}$$

$$\varphi_{12}(s) = \left.\frac{X_1(s)}{x_2(0)}\right|_{x_1(0)=0} = \frac{\frac{1}{s}(-2s^{-1})}{1+3s^{-1}+2s^{-2}} = \frac{-2}{(s^2+3s+2)}$$

$$\varphi_{21}(s) = \left.\frac{X_2(s)}{x_1(0)}\right|_{x_2(0)=0} = \frac{\frac{1}{s}(s^{-1})}{1+3s^{-1}+2s^{-2}} = \frac{1}{(s^2+3s+2)}$$

$$\varphi_{22}(s) = \left.\frac{X_2(s)}{x_2(0)}\right|_{x_1(0)=0} = \frac{\frac{1}{s}(1)}{1+3s^{-1}+2s^{-2}} = \frac{s}{(s^2+3s+2)}$$

Hence

$$\Phi(s) = \begin{bmatrix} \frac{s+3}{(s^2+3s+2)} & \frac{-2}{(s^2+3s+2)} \\ \frac{1}{(s^2+3s+2)} & \frac{s}{(s^2+3s+2)} \end{bmatrix}$$

Same answer as (1)

Comments

- We can now find $\Phi(t)$ if we wish

$$\Phi(t) = \mathcal{L}^{-1}\Phi(s) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

- We can also find the states and the output for any initial conditions. For example when $x_1(0) = x_2(0) = 1$ and $u(t) = 0$, we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi(t)\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$
