1) Solution:

a) Derive the transfer function

**Method 1 (Laplace Transform approach)**

We have

$$\dot{x}_1(t) = -3x_1(t) + x_2(t) + u(t),$$
$$\dot{x}_2(t) = -2x_2(t) + 2u(t),$$
$$y(t) = x_1(t) + x_2(t).$$

Taking the Laplace transforms of the above equations we get

$$[s+3]X_1(s) = X_2(s) + U(s),$$
$$[s+2]X_2(s) = 2U(s)$$
$$\Rightarrow X_2(s) = \frac{2}{s+2}U(s).$$

Therefore, we have

$$X_1(s) = \frac{2}{(s+3)(s+2)}U(s) + \frac{1}{s+3}U(s),$$
$$\Rightarrow X_1(s) = \frac{(s+4)}{(s+3)(s+2)}U(s).$$

Hence

$$Y(s) = X_1(s) + X_2(s) = \frac{s+4}{(s+3)(s+2)}U(s) + \frac{2}{s+2}$$
$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{(3s+10)}{(s+3)(s+2)}.$$

**Method 2 (State Space approach)**

The transfer function of the system can also be computed by using the formula given by

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D.$$

In this problem, we have

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$
$$D = 0.$$

Therefore, we have

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \left( \begin{bmatrix} s+3 & -1 \\ 0 & s+2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{(s+3)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$= \frac{3s+10}{(s+3)(s+2)}.$$

This coincides with the answer from method 1.

(b) The poles of the system are at –3 and –2. Since the poles of the system lie in the left half plane, the given system is stable.

2) Solution:

We first notice that

$$L[y(t)] = Y(s) = H(s)L[(u(t)] = H(s)\frac{1}{s}$$
$$\Rightarrow H(s) = sY(s)$$

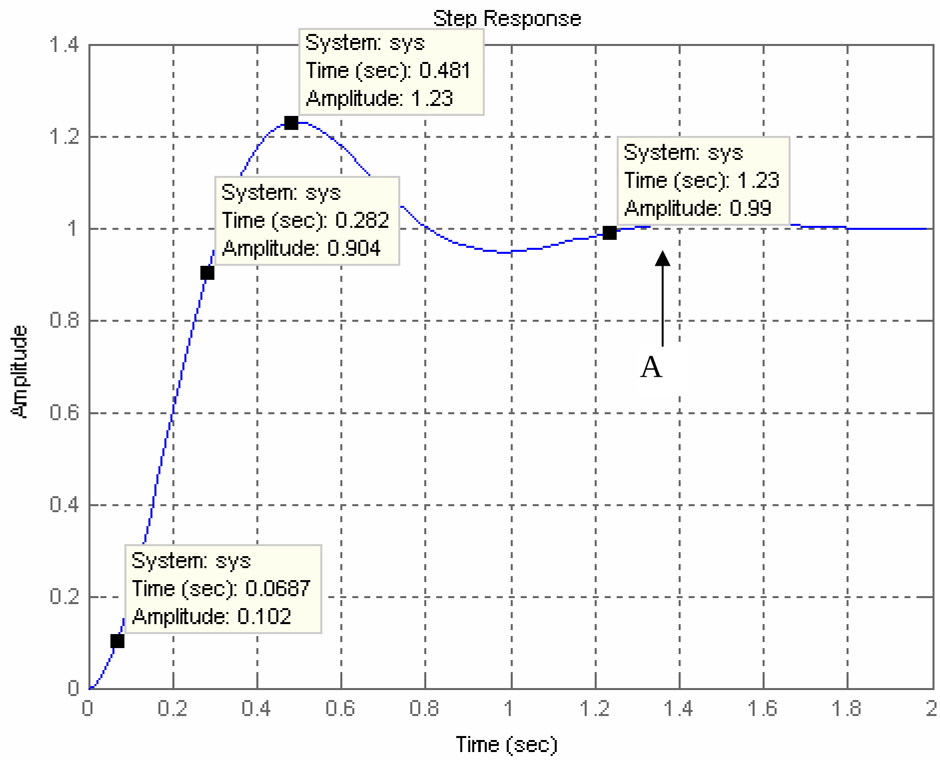
Hence

$$H(s) = s\left[\frac{5}{s} - \frac{2}{s+3} - \frac{3}{s+6}\right] = 6\frac{4s+15}{s(s+3)(s+6)}$$

Thus the system has three poles at s1=0, s2=-3, and s3=-6,

one zero located at s=-15/4.

3) Solution:




(a) The transfer function is given by:

$$TF = \frac{Y(s)}{R(s)} = \frac{k}{(s+4)(s+b)+kK} = \frac{k}{s^2+(4+b)s+4b+kK} \qquad (1)$$

Since *R(s)* = 1 / *s* (unit step input), the steady state value of the output is:

$$SS = \lim_{s\to 0}(TF) = \frac{k}{4b+kK} \qquad (2)$$

From the figure for the step response, *SS* = 1, so, equation (2) becomes:

$$\frac{k}{4b+kK} = 1 \Rightarrow k = 4b+kK \qquad (3)$$

Comparing equation (1) with the standard expression for the transfer function of a second order system gives

$$\omega_n^2 = k = 4b+kK \qquad \text{and} \qquad 2\xi\omega_n = 4+b \qquad (4)$$

From the figure for the step response, the peak time is given by:

$t_p = 0.49\ s$

The maximum overshoot is:

$$Mp = \frac{1.23 - 1}{1} = 0.23$$

The times for amplitude values of 10% and 90% *SS* can be obtained from the figure are approximately 0.08s and 0.355 s, respectively, so the rise time is:

$t_r$ = 0.282 - 0.0687 = 0.2133 *s*

With the maximum overshoot we get the damping factor:

$$\xi = -\frac{\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}} = -\frac{\ln 0.23}{\sqrt{\pi^2 + (\ln 0.23)^2}} = 0.4237$$

The damped frequency is related to the peak time, the damping factor, and the natural frequency by:

$$\omega_d = \frac{\pi}{t_p} = \omega_n \sqrt{1 - \xi^2} \rightarrow \omega_n = \frac{\pi}{t_p \sqrt{1 - \xi^2}} \rightarrow \omega_n = \frac{\pi}{0.49\sqrt{1 - 0.4237^2}} = 7.074 \text{ rad/s}$$

With equations (4) we obtain *k* and *b*:

$$k = \omega_n^2 = (7.074)^2 = 50\, rad^2/s^2$$

$$b = 2\xi\omega_n - 4 = 2$$

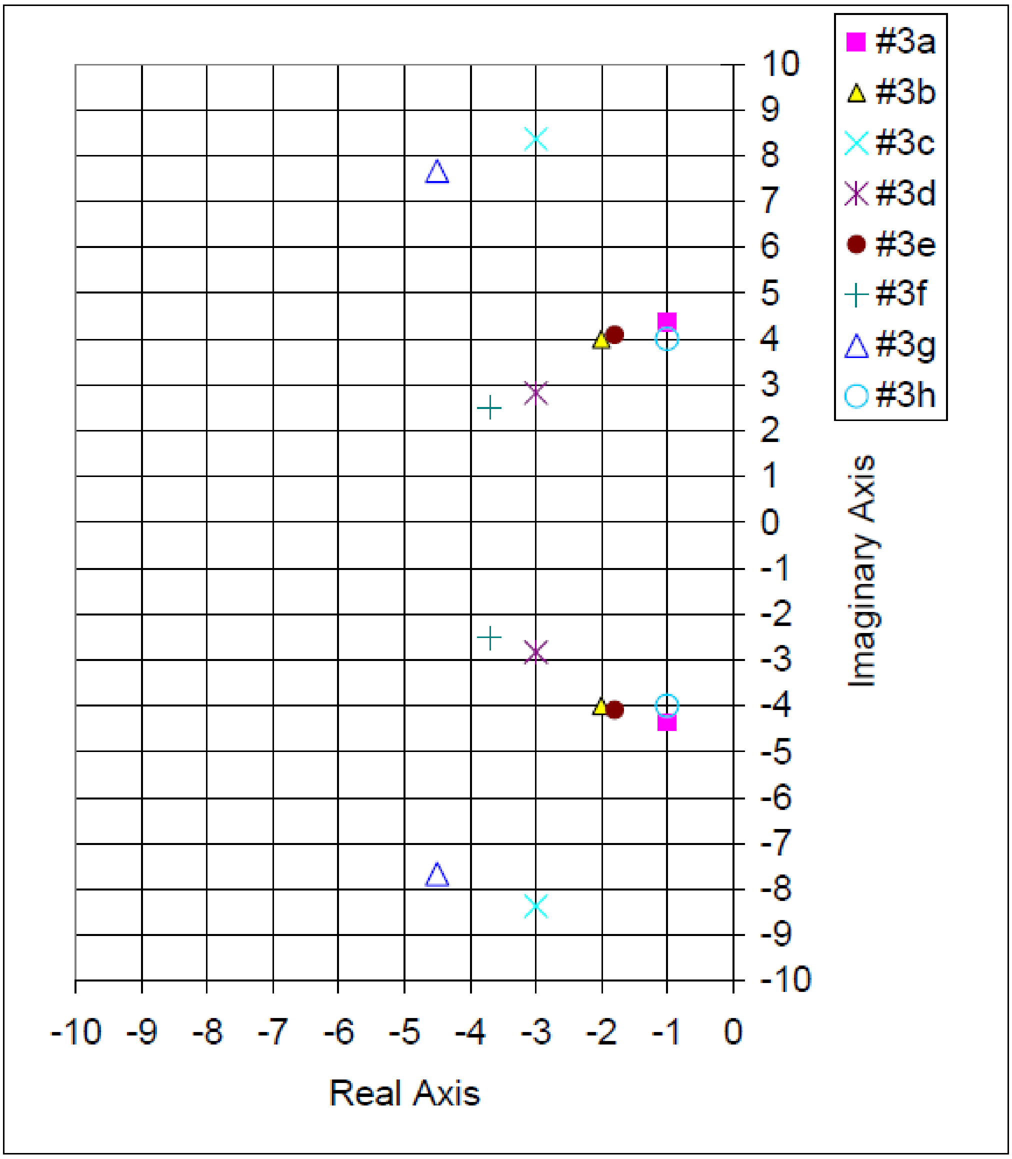
$$K = 1 - \frac{4b}{k} = 1 - \frac{8}{50} = 0.84$$

(b) On the figure for the step response, point (A) gives the position where the amplitude reachs 99% of *SS*, therefore the settling time for 1% criterion is:

$t_s$=1.23s

Problem 4

| Problem | $2\zeta\omega_n$ | $\omega_n^2$ | $\zeta$ | $\omega_n$ | Real | Imaginary |
|---|---|---|---|---|---|---|
| #3a | 2 | 20 | 0.224 | 4.47 | -1 | 4.36 |
| #3b | 4 | 20 | 0.447 | 4.47 | -2 | 4.00 |
| #3c | 6 | 79 | 0.338 | 8.89 | -3 | 8.37 |
| #3d | 6 | 17 | 0.728 | 4.12 | -3 | 2.83 |
| #3e | 3.6 | 20 | 0.402 | 4.47 | -1.8 | 4.09 |
| #3f | 7.4 | 20 | 0.827 | 4.47 | -3.7 | 2.51 |
| #3g | 9 | 79 | 0.506 | 8.89 | -4.5 | 7.66 |
| #3h | 2 | 17 | 0.243 | 4.12 | -1 | 4.00 |



**E5.2** (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{100}{(s+2)(s+5)+100} = \frac{100}{s^2+2\zeta\omega_n s+\omega_n^2} .$$

The steady-state error is given by

$$e_{ss} = \frac{A}{1+K_p} ,$$

where $R(s) = A/s$ and

$$K_p = \lim_{s\to 0} G(s) = \frac{100}{10} = 10 .$$

Therefore,

$$e_{ss} = \frac{A}{11} .$$

(b) The closed-loop system is a second-order system with natural frequency

$$\omega_n = \sqrt{110} ,$$

and damping ratio

$$\zeta = \frac{7}{2\sqrt{110}} = 0.334 .$$

Since the steady-state value of the output is 0.909, we must modify the percent overshoot formula which implicitly assumes that the steady-state value is 1. This requires that we scale the formula by 0.909. The percent overshoot is thus computed to be

$$P.O. = 0.909(100e^{-\pi\zeta/\sqrt{1-\zeta^2}}) = 29\% .$$

**E5.5** (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+GH(s)} = \frac{100}{s^2+100Ks+100} ,$$

where $H(s) = 1 + Ks$ and $G(s) = 100/s^2$. The steady-state error is computed as follows:

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s[R(s) - Y(s)] = \lim_{s\to 0} s[1 - T(s)]\frac{A}{s^2} \\ &= \lim_{s\to 0} \left[ 1 - \frac{\frac{100}{s^2}}{1+\frac{100}{s^2}(1+Ks)} \right] \frac{A}{s} = KA . \end{aligned}$$

(b) From the closed-loop transfer function, $T(s)$, we determine that $\omega_n = 10$ and

$$\zeta = \frac{100K}{2(10)} = 5K .$$

We want to choose $K$ so that the system is critically damped, or $\zeta = 1.0$. Thus,

$$K = \frac{1}{5} = 0.20 .$$

The closed-loop system has no zeros and the poles are at

$$s_{1,2} = -50K \pm 10\sqrt{25K^2 - 1} .$$

The percent overshoot to a step input is

$$P.O. = 100e^{\frac{-5\pi K}{\sqrt{1-25K^2}}} \quad \text{for} \quad 0 < K < 0.2$$

and $P.O. = 0$ for $K > 0.2$.

**E5.9** The second-order closed-loop transfer function is given by

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \ .$$

From the percent overshoot specification, we determine that

$$P.O. \leq 5\% \quad \text{implies} \quad \zeta \geq 0.69 \ .$$

From the settling time specification, we find that

$$T_s < 4 \quad \text{implies} \quad \omega_n\zeta > 1 \ .$$

And finally, from the peak time specification we have

$$T_p < 1 \quad \text{implies} \quad \omega_n\sqrt{1-\zeta^2} > \pi \ .$$

The constraints imposed on $\zeta$ and $\omega_n$ by the performance specifications define the permissible area for the poles of $T(s)$, as shown in Figure E5.9.

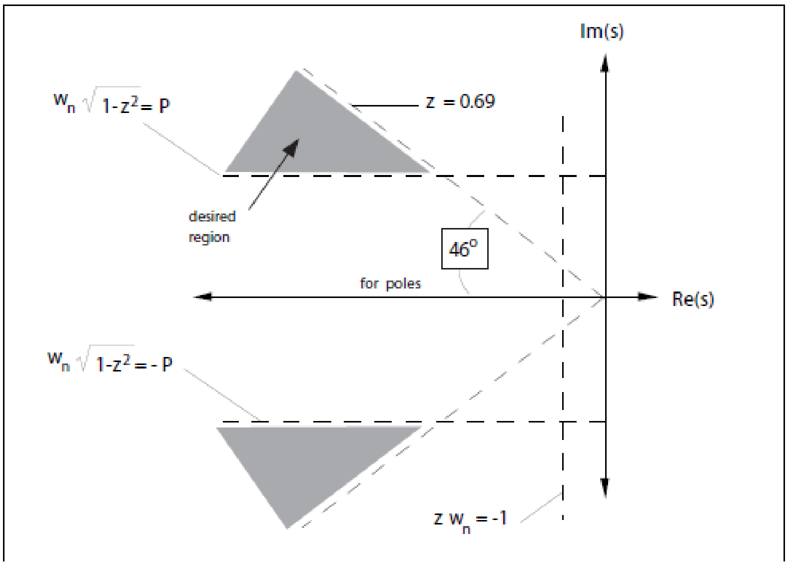



**FIGURE E5.9**
Permissible area for poles of $T(s)$.

**E5.17** The output is given by

$$Y(s) = T(s)R(s) = K\frac{G(s)}{1+G(s)}R(s) .$$

When $K = 1$, the steady-state error is

$$e_{ss} = 0.2$$

which implies that

$$\lim_{s\to 0} sY(s) = 0.8 .$$

Since we want $e_{ss} = 0$, it follows that

$$\lim_{s\to 0} sY(s) = 1 ,$$

or

$$0.8K = 1 .$$

Therefore, $K = 1.25$.

**P5.2** (a) The settling time specification

$$T_s = \frac{4}{\zeta\omega_n} < 0.6$$

is used to determine that $\zeta\omega_n > 6.67$. The *P.O.* $< 20\%$ requirement is used to determine

$$\zeta < 0.45 \quad \text{which implies} \quad \theta < 63^o$$

and the *P.O.* $> 10\%$ requirement is used to determine

$$\zeta > 0.60 \quad \text{which implies} \quad \theta > 53^o,$$

since $\cos\theta = \zeta$. The desired region for the poles is shown in Figure P5.2.

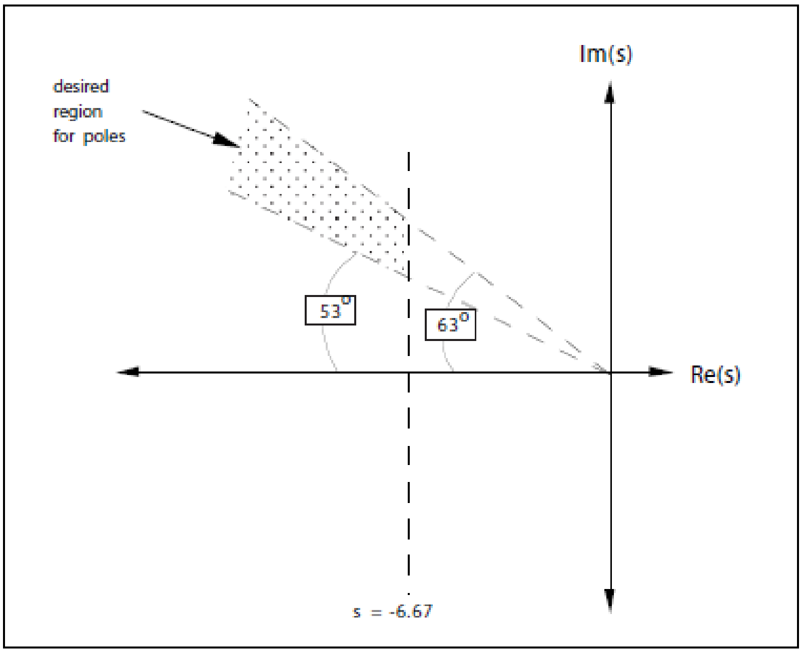



**FIGURE P5.2**
Desired region for pole placement.

(b) The third root should be at least 10 times farther in the left half-plane, so

$$|r_3| \geq 10|\zeta\omega_n| = 66.7 \ .$$

(c) We select the third pole such that $r_3 = -66.7$. Then, with $\zeta = 0.45$ and $\zeta\omega_n = 6.67$, we determine that $\omega_n = 14.8$. So, the closed-loop transfer function is

$$T(s) = \frac{66.7(219.7)}{(s + 66.7)(s^2 + 13.3s + 219.7)} \ ,$$

where the gain $K = (66.7)(219.7)$ is chosen so that the steady-state

tracking error due to a step input is zero. Then,

$$T(s) = \frac{G(s)}{1 + G(s)} \ ,$$

or

$$G(s) = \frac{T(s)}{1 - T(s)} \ .$$

**P5.10** (a) The armature controlled DC motorblock diagram is shown in Figure P5.10.

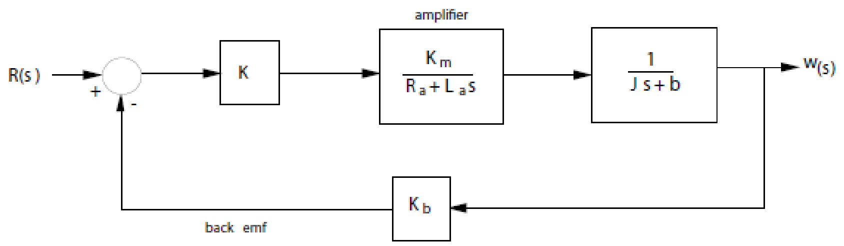



**FIGURE P5.10**
Armature controlled DC motor block diagram.

(b) The closed-loop transfer function is

$$T(s) = \frac{\omega(s)}{R(s)} = \frac{KG(s)}{1 + KK_bG(s)} ,$$

where

$$G(s) = \frac{K_m}{(R_a + L_a s)(Js + b)} .$$

Thus,

$$T(s) = \frac{K}{s^2 + 2s + 1 + K} ,$$

where $R_a = L_a = J = b = K_b = K_m = 1$. The steady-state tracking error is

$$e_{ss} = \lim_{s\to 0} s(R(s) - Y(s)) = \lim_{s\to 0} s\left(\frac{A}{s}\right)(1 - T(s))$$
$$= A(1 - T(0)) = \left(1 - \frac{K}{1+K}\right) = \frac{A}{1+K} .$$

(c) For a percent overshoot of 15%, we determine that $\zeta = 0.5$. From our characteristic polynomial we have $2\zeta\omega_n = 2$ and $\omega_n = \sqrt{1+K}$. Solving for $\omega_n$ yields $\omega_n = 2$, thus $K = 3$.

**AP5.7** The performance is summarized in Table AP5.7 and shown in graphical form in Fig. AP5.7.

| $K$ | Estimated Percent Overshoot | Actual Percent Overshoot |
|---|---|---|
| 1000 | 8.8 % | 8.5 % |
| 2000 | 32.1 % | 30.2 % |
| 3000 | 50.0 % | 46.6 % |
| 4000 | 64.4 % | 59.4 % |
| 5000 | 76.4 % | 69.9 % |

**TABLE AP5.7** Performance summary.

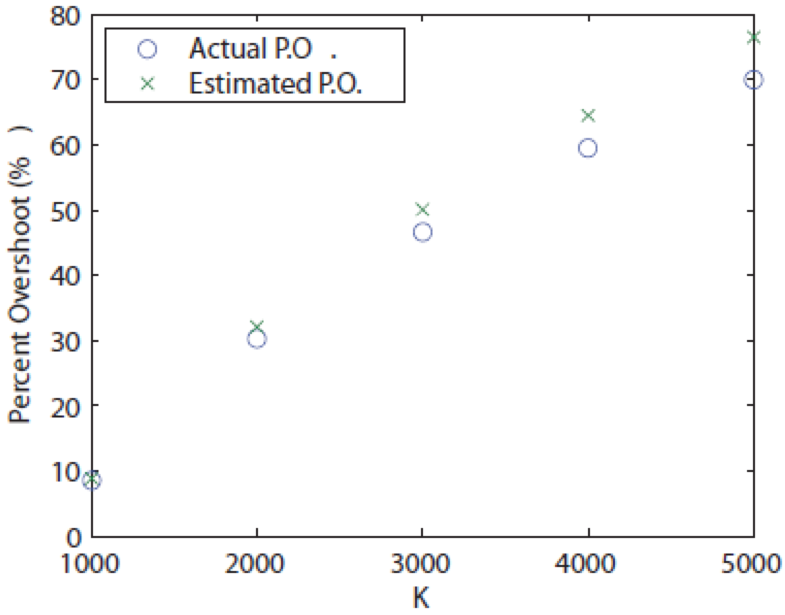



**FIGURE AP5.7**
Percent overshoot versus K.

The closed-loop transfer function is

$$T(s) = \frac{100K}{s(s+50)(s+100)+100K} \ .$$

The impact of the third pole is more evident as $K$ gets larger as the estimated and actual percent overshoot deviate in the range 0.3% at $K = 1000$ to 6.5% at $K = 5000$.