Problem 4.18

Using Gauss's law in differential form: $\nabla\vec{D} = \rho_v$

a) In rectangular coordinates $\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad \Rightarrow \quad \rho_v = 8y + 0 = 8y$

b) In cylindrical coordinates $\rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$

$$\Rightarrow \quad \rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}(4\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial(2\rho\cos\phi)}{\partial \phi} + \frac{\partial(2z^2)}{\partial z} \quad \Rightarrow \quad \rho_v = 8\sin\phi - 2\sin\phi + 4z$$

c) In spherical coordinates $\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$

$$\Rightarrow \quad \rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{2\cos\theta}{r^3}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\frac{\sin\theta}{r^3}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial 0}{\partial \phi}$$

$$= \frac{1}{r^2}2\cos\theta(-r^{-2}) + \frac{1}{r^4\sin\theta}(2\sin\theta\cos\theta)$$

$$= -\frac{2\cos\theta}{r^4} + \frac{2\cos\theta}{r^4}$$

$$\therefore \quad \rho_v = 0$$

Problem 4.22

a) $\rho_v = \dfrac{\partial(2xy)}{\partial x} + \dfrac{\partial(x^2)}{\partial y} + \dfrac{\partial(0)}{\partial z} = 2y$

b) $\psi = \int_S \vec{D}.\vec{ds} \quad \Rightarrow \quad \psi = \int_S \vec{D}.(dxdz\vec{a}_y) = \int_S D_y dxdz = \int_{x=0}^{1}\int_{z=0}^{1} x^2 dxdz$

$\therefore \quad \psi = \dfrac{1}{3} C/m^3$

c) Method 1: Using $Q = \int_V \rho_v \, dv$

$Q = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2y)dxdydz = 1C$

Method 2: Using Gauss's law in integral form $Q = \oint_S \vec{D}.\vec{ds}$

$$Q = \int_{top} \vec{D}.\vec{ds} + \int_{bottom} \vec{D}.\vec{ds} + \int_{left} \vec{D}.\vec{ds} + \int_{right} \vec{D}.\vec{ds} + \int_{back} \vec{D}.\vec{ds} + \int_{front} \vec{D}.\vec{ds}$$

$$= \int_{top} \vec{D}.(dxdy\vec{a}_z) + \int_{bottom} \vec{D}.(-dxdy\vec{a}_z) + \int_{left} \vec{D}.(-dxdz\vec{a}_y) + \int_{right} \vec{D}.(dxdz\vec{a}_y) + \int_{back} \vec{D}.(-dydz\vec{a}_x) + \int_{front} \vec{D}.(dydz\vec{a}_x)$$

Since $\vec{D}$ does not have a z-component, then the surface integral over the "top" and "bottom" surfaces give zeros. Therefore,

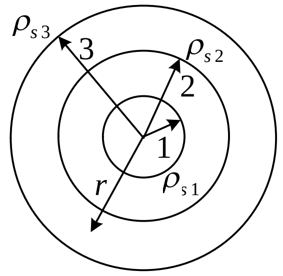
$$Q = -\int_{left} x^2 dxdz + \int_{right} x^2 dxdz - \int_{back} 2xydydz + \int_{front} 2xydydz$$

$$= -\int_{x=0}^{1}\int_{z=0}^{1} x^2 dxdz\Big|_{y=0} + \int_{x=0}^{1}\int_{z=0}^{1} x^2 dxdz\Big|_{y=1} - \int_{z=0}^{1}\int_{y=0}^{1} 2xydydz\Big|_{x=0} + \int_{z=0}^{1}\int_{y=0}^{1} 2xydydz\Big|_{x=1}$$

$$= -\frac{1}{3} + \frac{1}{3} - 0 + 1 = 1$$

$\therefore \quad Q = 1C$ (the same answer as in method 1).

Problem 4.25



a) Using Gauss's law in integral form $\psi = \oint_S \vec{D}.\vec{ds} = Q$

The electric flux through the closed surface $r = 1.5 \quad \Rightarrow \quad \psi = Q = \rho_{s1} S_1 = 2\times10^{-6}(4\pi\times1^2) = 8\pi10^{-6}$ C

The flux through $r = 2.5 \quad \Rightarrow \quad \psi = Q = \rho_{s1} S_1 + \rho_{s2} S_2 = 2\times10^{-6}(4\pi\times1^2) - 4\times10^{-6}(4\pi\times2^2) = -56\pi10^{-6}$ C

b) From the symmetry of the problem, we conclude that $\vec{D}$ has only a radial component. Thus $\vec{D} = D_r \vec{a}_r$.

Using $\oint_S \vec{D}.\vec{ds} = Q \quad \Rightarrow \quad D_r 4\pi r^2 = Q \quad \Rightarrow \quad \therefore \vec{D} = \frac{Q}{4\pi r^2}\vec{a}_r$

At $r = 0.5 \quad \Rightarrow \quad \vec{D} = \frac{0}{4\pi(0.5)^2}\vec{a}_r = \vec{0}$

At $r = 2.5 \quad \Rightarrow \quad \vec{D} = \frac{\rho_{s1}S_1 + \rho_{s2}S_2}{4\pi(2.5)^2}\vec{a}_r = \frac{-56\pi10^{-6}}{4\pi(2.5)^2}\vec{a}_r = -2.24\vec{a}_r$ C/m^2

At $r = 3.5 \quad \Rightarrow \quad \vec{D} = \frac{\rho_{s1}S_1 + \rho_{s2}S_2 + \rho_{s3}S_3}{4\pi(3.5)^2}\vec{a}_r$

$$\vec{D} = \frac{2\times10^{-6}(4\pi\times1^2) - 4\times10^{-6}(4\pi\times2^2) + 5\times10^{-6}(4\pi\times3^2)}{4\pi(3.5)^2}\vec{a}_r$$

$$\vec{D} = \frac{124\pi10^{-6}}{4\pi(3.5)^2}\vec{a}_r = 2.531\vec{a}_r \text{ C/m}^2$$

Problem 4.27

Using Gauss's law in integral form $\oint_S \vec{D}.\vec{ds} = Q$

From the problem symmetry, we conclude that $\vec{D}$ has a component along the ρ direction only. Thus $\vec{D} = D_\rho \vec{a}_\rho$.

Using a *closed* cylindrical surface S with a *variable* radius ρ and length h :

$$\oint_S \vec{D}.\vec{ds} = Q \quad \Rightarrow \quad D_\rho 2\pi\rho h = Q \quad \Rightarrow \quad D_\rho = \frac{Q}{2\pi\rho h} \quad \Rightarrow \quad \vec{D} = \frac{Q}{2\pi\rho h}\vec{a}_\rho$$

For $1 > \rho \ge 0$, $\vec{D} = \frac{Q}{2\pi\rho h}\vec{a}_\rho \quad \Rightarrow \quad \vec{D} = \frac{0}{2\pi\rho h}\vec{a}_\rho = \vec{0}$

For $2 > \rho > 1$, $\vec{D} = \frac{Q}{2\pi\rho h}\vec{a}_\rho \quad \Rightarrow \quad \vec{D} = \frac{\int_V \rho_v dv}{2\pi\rho h}\vec{a}_\rho = \frac{\int_{\rho=1}^{\rho}\int_{\phi=0}^{2\pi}\int_{z=0}^{h}(12\rho\times10^{-9})(\rho d\rho d\phi dz)}{2\pi\rho h}\vec{a}_\rho$

$$\vec{D} = \frac{12\times10^{-9}\times2\pi h\times\left.\frac{\rho^3}{3}\right|_1^{\rho}}{2\pi\rho h}\vec{a}_\rho = \frac{4\times10^{-9}(\rho^3-1)}{\rho}\vec{a}_\rho$$

For $\rho > 2$, $\vec{D} = \frac{Q}{2\pi\rho h}\vec{a}_\rho \quad \Rightarrow \quad \vec{D} = \frac{\int_V \rho_v dv}{2\pi\rho h}\vec{a}_\rho = \frac{\int_{\rho=1}^{2}\int_{\phi=0}^{2\pi}\int_{z=0}^{h}(12\rho\times10^{-9})(\rho d\rho d\phi dz)}{2\pi\rho h}\vec{a}_\rho$

$$\vec{D} = \frac{12\times10^{-9}\times2\pi h\times\left.\frac{\rho^3}{3}\right|_1^{2}}{2\pi\rho h}\vec{a}_\rho = \frac{4\times10^{-9}(8-1)}{\rho}\vec{a}_\rho = \frac{28\times10^{-9}}{\rho}\vec{a}_\rho$$

Problem 4.41

$$V_{ab} = -\int_a^b \vec{E}.\vec{dl} \quad \Rightarrow \quad W_{ab} = -q\int_a^b \vec{E}.\vec{dl}$$

a) $W_{AB} = -q\int_A^B \vec{E}.\vec{dl} = -4\times10^{-9}\int_{\rho=1}^{4} E_\rho\Big|_{\phi=0,\, z=0} d\rho = -4\times10^{-9}\int_{\rho=1}^{4} 0 d\rho = 0$

b) $W_{BC} = -q\int_B^C \vec{E}.\vec{dl} = -4\times10^{-9}\int_{\phi=0^o}^{30^o} E_\phi \rho d\phi\Big|_{\rho=4, z=0} = -4\times10^{-9}\int_{\phi=0^o}^{30^o} 4\cos\phi\times4d\phi = -32\times10^{-9} = -32\ nJ$

c) $W_{CD} = -q\int_C^D \vec{E}.\vec{dl} = -4\times10^{-9}\int_{z=0}^{-2} E_z\Big|_{\rho=4,\, \phi=30^o} dz = -4\times10^{-9}\int_{z=0}^{-2} 4\sin30^o dz = 16\times10^{-9} = 16\text{ nJ}$

d) $W_{AD} = (0-32+16)\times10^{-9} = -16\text{ nJ}$

Problem 4.43

$\vec{E} = -\nabla V$

a) In rectangular coordinates $\vec{E} = -(\frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z)$

For $V = x^2 + 2y^2 + 4z^2 \quad \Rightarrow \quad \vec{E} = -(2x\vec{a}_x + 4y\vec{a}_y + 8z\vec{a}_z)$ V/m

d) In spherical coordinates $\vec{E} = -(\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi)$

For $V = e^{-r}\sin\theta\cos 2\phi \quad \Rightarrow \quad \vec{E} = -(-e^{-r}\sin\theta\cos 2\phi\vec{a}_r + \frac{e^{-r}\cos\theta\cos 2\phi}{r}\vec{a}_\theta + \frac{-2e^{-r}\sin\theta\sin 2\phi}{r\sin\theta}\vec{a}_\phi)$

$$\vec{E} = e^{-r}\sin\theta\cos 2\phi\vec{a}_r - \frac{e^{-r}}{r}\cos\theta\cos 2\phi\vec{a}_\theta + \frac{2e^{-r}}{r}\sin 2\phi\vec{a}_\phi$$