King Fahd University of Petroleum & Minerals
Department of Electrical Engineering

Communications Engineering I
EE 370

Course Notes
Chapter 3

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Chapter 3 Analysis and transmission of signals

In the last chapter we discussed spectral representation of periodic signals (Fourier series). In this chapter we extend this spectral representation to aperiodic signals.

1. Aperiodic signal representation by Fourier integral:

To represent an aperiodic signal $g(t)$, such as the one shown in Fig. 3.1.a by everlasting exponential signals, let us construct a new periodic signal $g_{T_0}(t)$ formed by repeating the signal $g(t)$ every T_0 seconds, as shown in Fig. 3.1b. The periodic signal $g_{T_0}(t)$ can be represented by an exponential Fourier series. If we let $T_0 \to \infty$, the pulses in the periodic signal repeat after an infinite interval, and therefore

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)$$

Thus, the Fourier series representing $g_{T_0}(t)$ will also represent $g(t)$ in the limit $T_0 \to \infty$. The exponential Fourier series for $g_{T_0}(t)$ is:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where $D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-jn\omega_0 t} dt$, $\omega_0 T_0 = 2\pi$

Observe that integrating $g_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. Therefore, D_n can be expressed as:

$$D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} g(t) e^{-jn\omega_0 t} dt$$

which can be set up as: $D_n = \frac{1}{T_0} G(n\omega_0)$

with $G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$

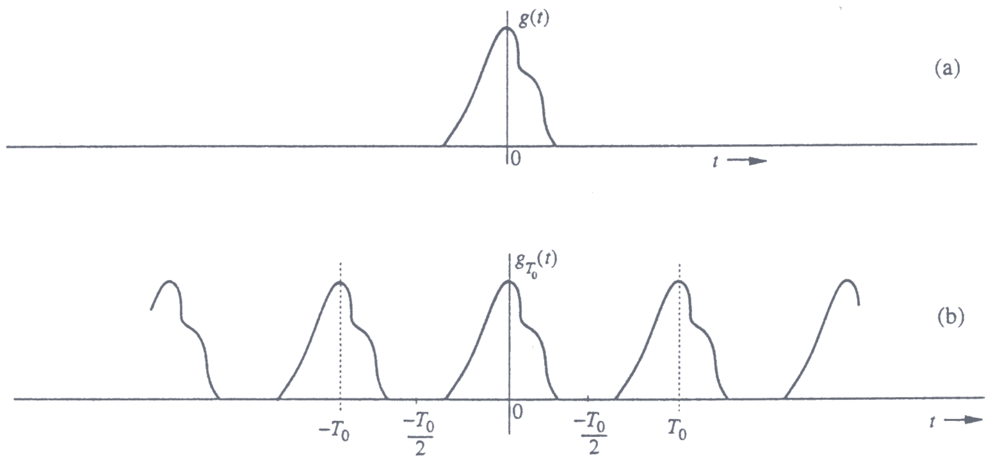


Figure 3.1 Construction of a periodic signal by periodic extension of $g(t)$.

The Fourier integral (The Fourier series becomes the Fourier integral in the limit as $T_0 \to \infty$) is given by:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$$

We call $G(\omega)$ the direct Fourier transform of $g(t)$, and $g(t)$ the inverse Fourier transform of $G(\omega)$.

$\therefore$ $g(t)$ and $G(\omega)$ are a Fourier transform pair. Symbolically, this is expressed as:

$$G(\omega) = \mathcal{F}[g(t)] \quad \text{and} \quad g(t) = \mathcal{F}^{-1}[G(\omega)]$$

or

$$g(t) \Longleftrightarrow G(\omega)$$

To recapitulate,

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

and

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$$

It is helpful to keep in mind that the Fourier integral $\left(g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega\right)$ is of the nature of a Fourier series with fundamental frequency $\Delta\omega \to 0$. Therefore, most of the discussion and properties of Fourier series apply to the Fourier transform as well. We can plot the spectrum $G(\omega)$ as a function of ω. Since $G(\omega)$ is complex, we have both amplitude and angle (or phase) spectra:

$$G(\omega) = |G(\omega)|\, e^{j\theta_g(\omega)}$$

where $|G(\omega)|$ is the amplitude and $\theta_g(\omega)$ is the angle (or phase) of $G(\omega)$.

* <u>Conjugate symmetry property</u>:

$$G(-\omega) = \int_{-\infty}^{\infty} g(t)\, e^{j\omega t}\, dt$$

From the above equation, it follows that if $g(t)$ is a real function of t, then $G(\omega)$ and $G(-\omega)$ are complex conjugates, that is,

$$G(-\omega) = G^*(\omega)$$

Therefore,

$$|G(-\omega)| = |G(\omega)|$$
$$\theta_g(-\omega) = -\theta_g(\omega)$$

Thus, for real $g(t)$, the amplitude spectrum $|G(\omega)|$ is an even function of ω, and the phase spectrum $\theta_g(\omega)$ is an odd function of ω.

Example: Find the Fourier transform of $e^{-at}u(t)$.

* <u>Linearity of the Fourier transform:</u>

$$g_1(t) \Longleftrightarrow G_1(\omega) \quad \text{and} \quad g_2(t) \Longleftrightarrow G_2(\omega)$$

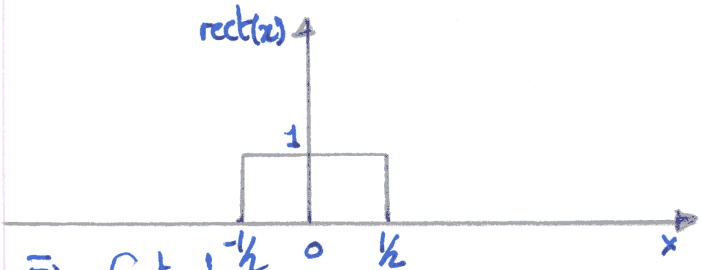
then

$$a_1 g_1(t) + a_2 g_2(t) \Longleftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$$

2. Transforms of some useful functions:

* <u>Unit Gate function</u>

We define a unit gate function $\mathrm{rect}(x)$ as a gate of unit height and unit width, centered at the origin:



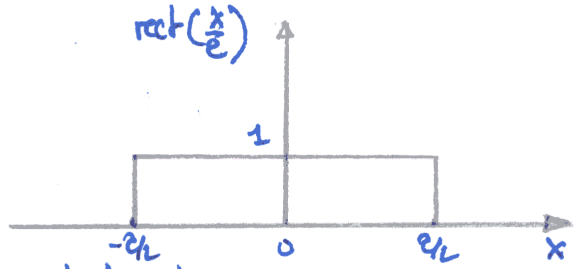


Fig. Gate pulse.

$$\mathrm{rect}(x) = \begin{cases} 0 & , |x| > \frac{1}{2} \\ \frac{1}{2} & , |x| = \frac{1}{2} \\ 1 & , |x| < \frac{1}{2} \end{cases}$$

Observe that τ, the denominator of the argument of rect $\left(\frac{x}{\tau}\right)$, indicates the width of the pulse in Fig. 3.7b.

* <u>Unit triangle function:</u>

It is defined as:

$$\Delta(x) = \begin{cases} 0 & , |x| > \frac{1}{2} \\ 1-2|x| & , |x| < \frac{1}{2} \end{cases}$$

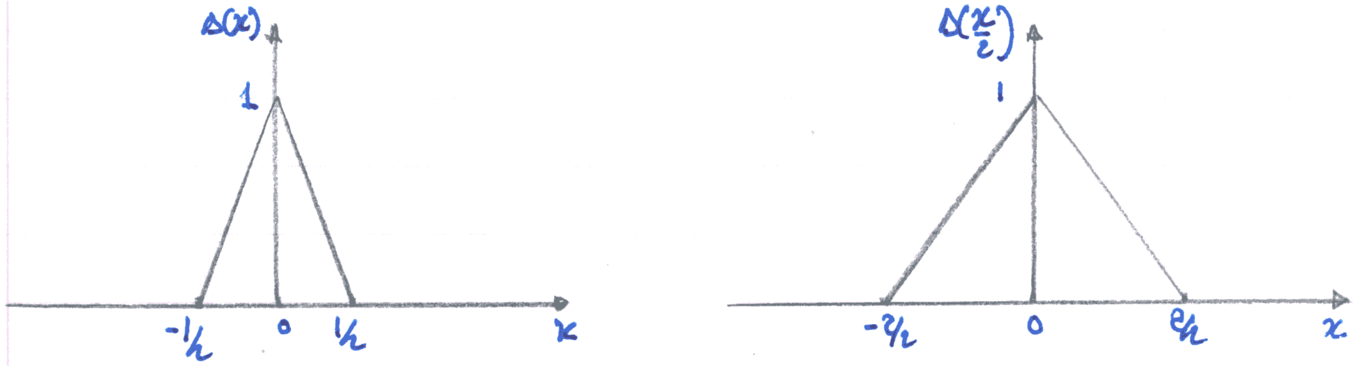


Fig. Triangle pulse.

* <u>Interpolation function sinc(x):</u>

The filtering or interpolating function is defined as:

$$\operatorname{sinc}(x) = \frac{\sin x}{x}$$

Observe that:

- $\operatorname{sinc}(x)$ is an even function of x.
- $\operatorname{sinc}(x) = 0$ for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
- $\operatorname{sinc}(0) = 1$ [use L'Hôpital's rule].

Example: Find the Fourier transform of $g(t) = \operatorname{rect}\left(\frac{t}{\tau}\right)$.

$$\operatorname{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

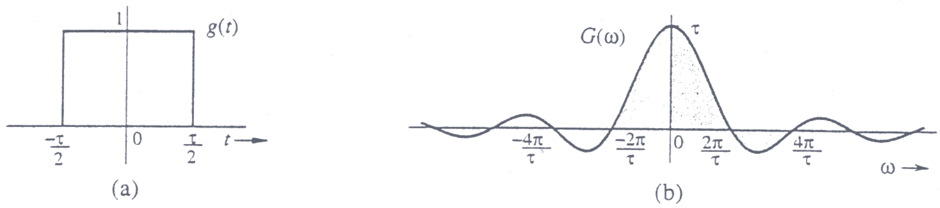


Figure 3.10 Gate pulse and its Fourier spectrum.

We have

$$G(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

Since rect $(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$\begin{aligned} G(\omega) &= \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt \\ &= -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin(\omega\tau/2)}{\omega} \\ &= \tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

Therefore,

$$\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right) \tag{3.17}$$

Recall that sinc $(x) = 0$ when $x = \pm n\pi$. Hence, sinc $(\omega\tau/2) = 0$ when $\omega\tau/2 = \pm n\pi$; that is, when $\omega = \pm 2n\pi/\tau$ $(n = 1, 2, 3, \cdots)$, as shown in Fig. 3.10b. Observe that in this case $G(\omega)$ happens to be real. Hence, we may convey the spectral information by a single plot of $G(\omega)$ shown in Fig. 3.10b.

EXAMPLE 3.3 Find the Fourier transform of the unit impulse $\delta(t)$.

Using the sampling property of the impulse [Eq. (2.19a)], we obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \tag{3.18a}$$

or

$$\delta(t) \Longleftrightarrow 1 \tag{3.18b}$$

Figure 3.11 shows $\delta(t)$ and its spectrum.

Figure 3.11 Unit impulse and its Fourier spectrum.

EXAMPLE 3.4 Find the inverse Fourier transform of $\delta(\omega)$.

From Eq. (3.8b) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}$$

Therefore,

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \tag{3.19a}$$

or

$$1 \Longleftrightarrow 2\pi\delta(\omega) \tag{3.19b}$$

This shows that the spectrum of a constant signal $g(t) = 1$ is an impulse $2\pi\delta(\omega)$, as shown in Fig. 3.12.

The result [Eq. (3.19b)] also could have been anticipated on qualitative grounds. Recall that the Fourier transform of $g(t)$ is a spectral representation of $g(t)$ in terms of everlasting exponential components of the form $e^{j\omega t}$. Now to represent a constant signal $g(t) = 1$, we need a single everlasting exponential* $e^{j\omega t}$ with $\omega = 0$. This results in a spectrum at a single frequency $\omega = 0$. Another way of looking at the situation is that $g(t) = 1$ is a dc signal which has a single frequency $\omega = 0$ (dc).

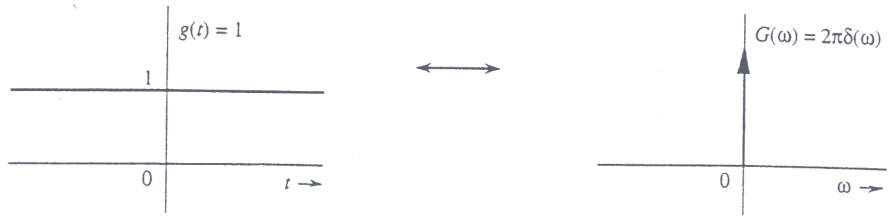


Figure 3.12 Constant (dc) signal and its Fourier spectrum.

If an impulse at $\omega = 0$ is a spectrum of a dc signal, what does an impulse at $\omega = \omega_0$ represent? We shall answer this question in the next example.

EXAMPLE 3.5 Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Therefore,

$$\frac{1}{2\pi} e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0) \tag{3.20a}$$

This result shows that the spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega = \omega_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting exponential $e^{j\omega_0 t}$, we need a single everlasting exponential $e^{j\omega t}$ with $\omega = \omega_0$. Therefore, the spectrum consists of a single component at frequency $\omega = \omega_0$.

From Eq. (3.20a) it follows that

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0) \tag{3.20b}$$

EXAMPLE 3.6 Find the Fourier transforms of the everlasting sinusoid $\cos \omega_0 t$.

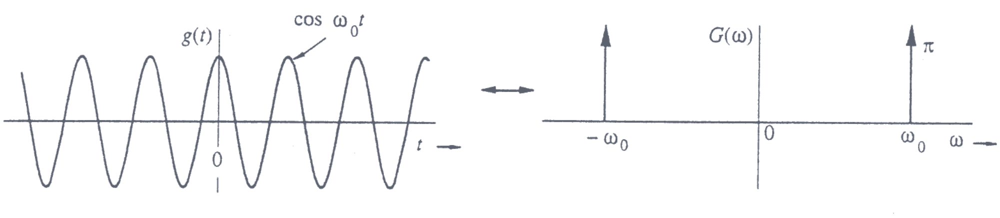


Figure 3.13 Cosine signal and its Fourier spectrum.

Recall the Euler formula

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Adding Eqs. (3.20a) and (3.20b), and using the above formula, we obtain

$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \tag{3.21}$$

The spectrum of $\cos \omega_0 t$ consists of two impulses at ω_0 and $-\omega_0$, as shown in Fig. 3.13. The result also follows from qualitative reasoning. An everlasting sinusoid $\cos \omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore, the Fourier spectrum consists of only two components of frequencies ω_0 and $-\omega_0$.

EXAMPLE 3.7 Find the Fourier transform of the sign function sgn t (pronounced signum t), shown in Fig. 3.14. Its value is $+1$ or -1, depending on whether t is positive or negative:

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} \tag{3.22}$$

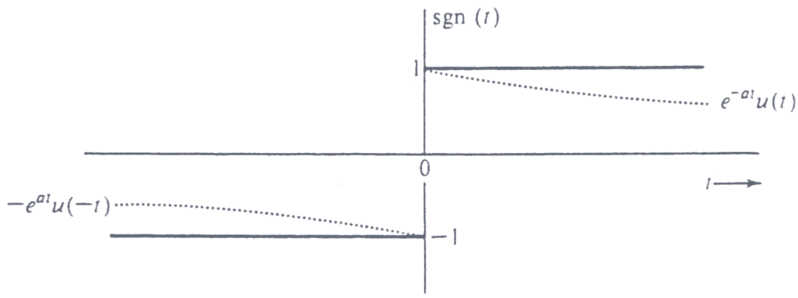


Figure 3.14 Sign function.

The transform of sgn t can be obtained by considering sgn t as a sum of two exponentials, as shown in Fig. 3.14, in the limit as $a \to 0$:

$$\operatorname{sgn} t = \lim_{a\to 0} \left[e^{-at}u(t) - e^{at}u(-t)\right]$$

Therefore,

$$\begin{aligned}
\mathcal{F}[\operatorname{sgn} t] &= \lim_{a\to 0} \left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\} \\
&= \lim_{a\to 0} \left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right) \qquad \text{(see pairs 1 and 2 in Table 3.1)} \\
&= \lim_{a\to 0} \left(\frac{-2j\omega}{a^2+\omega^2}\right) = \frac{2}{j\omega}
\end{aligned} \tag{3.23}$$

Example: Find the Fourier transform of the unit impulse $\delta(t)$.

$$\delta(t) \Longleftrightarrow 1$$

Example: Find the Fourier transform of $\cos \omega_0 t$.

$$\cos \omega_0 t \Longleftrightarrow \pi \left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right]$$

Example: Find the Fourier transform of the signum function defined as:

$$\operatorname{sgn} t = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

which can be obtained by considering:

$$\operatorname{sgn} t = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

Therefore, $\mathcal{F}[\operatorname{sgn} t] = \dfrac{2}{j\omega}$.

3. Some properties of the Fourier transform:

We now study some of the important properties of the Fourier transform and their implications as well as their applications.

3.1 Symmetry of Direct and Inverse transform Operations – Time-Frequency Duality

$$\begin{cases} G(\omega) = \displaystyle\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \\ g(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega \end{cases}$$

In these 2 operations there are only 2 minor differences: the factor 2π and the exponential indices in the 2 operations have opposite signs. Otherwise the 2 operations are symmetrical. This is the basis (this observation) of the so-called duality of time and frequency.

Example, the time-shifting property, states that if

$$g(t) \Leftrightarrow G(\omega) \quad , \text{ then } g(t-t_0) \Leftrightarrow G(\omega)\, e^{-j\omega t_0}$$

The dual property (the frequency-shifting property) states that

$$g(t)\, e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0) \, .$$

3.2 Symmetry property:

If $$g(t) \Leftrightarrow G(\omega)$$

then $$G(t) \Leftrightarrow 2\pi\, g(-\omega)$$

Proof: $$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x)\, e^{jxt}\, dx$$

$$2\pi\, g(-t) = \int_{-\infty}^{\infty} G(x)\, e^{-jxt}\, dx$$

Let $\omega = t$

$$\therefore \; 2\pi\, g(-\omega) = \int_{-\infty}^{\infty} G(x)\, e^{-j\omega x}\, dx$$

3.3 Scaling property:

If $$g(t) \Leftrightarrow G(\omega)$$

then, for any real constant a, $$g(at) \Leftrightarrow \frac{1}{|a|}\, G\!\left(\frac{\omega}{a}\right)$$

Proof: $$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at)\, e^{-j\omega t}\, dt = \frac{1}{a} \int_{-\infty}^{\infty} g(x)\, e^{-j\left(\frac{\omega}{a}\right)x}\, dx$$

$$= \frac{1}{a}\, G\!\left(\frac{\omega}{a}\right) \quad , \; a>0$$

Similarly, it can be show that if $a<0$

$$g(at) \Leftrightarrow -\frac{1}{a}\, G\!\left(\frac{\omega}{a}\right)$$

3.4 Time-shifting property:

If $$g(t) \Leftrightarrow G(\omega)$$

then $\quad g(t-t_0) \Leftrightarrow G(\omega) e^{-j\omega t_0}$

Proof:

$$\mathcal{F}[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)\, e^{-j\omega t}\, dt$$

let $t - t_0 = x$, we have

$$\mathcal{F}[g(t-t_0)] = \int_{-\infty}^{+\infty} g(x)\, e^{-j\omega(x+t_0)}\, dx$$

$$= e^{-j\omega t_0} \int_{-\infty}^{+\infty} g(x)\, e^{-j\omega x}\, dx$$

$$= G(\omega)\, e^{-j\omega t_0}$$

This result shows that delaying a signal by t_0 seconds does not change its amplitude spectrum. The phase spectrum, however, is changed by $-\omega t_0$.

3.5 Frequency-shifting property:

If $\quad g(t) \Longleftrightarrow G(\omega)$

then $\quad g(t)\, e^{j\omega_0 t} \Longleftrightarrow G(\omega - \omega_0)$

Proof:

$$\mathcal{F}[g(t)\, e^{j\omega_0 t}] = \int_{-\infty}^{\infty} g(t)\, e^{j\omega_0 t} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} g(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= G(\omega - \omega_0)$$

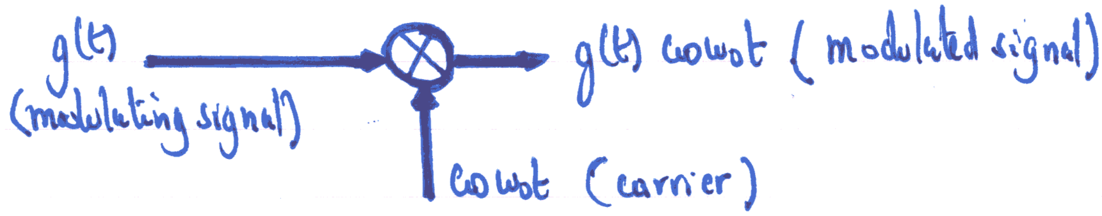
This property states that multiplication of a signal by a factor $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$. Note the duality between the time-shifting and the frequency-shifting properties.

Because $e^{j\omega_0 t}$ is not a real function that can be generated, frequency shifting in practice is achieved by multiplying $g(t)$ by a sinusoid. This can be seen from the fact that

it follows that
$$g(t)\cos\omega_0 t = \frac{1}{2}\left[g(t)e^{j\omega_0 t} + g(t)e^{-j\omega_0 t}\right]$$

$$g(t)\cos\omega_0 t \Longleftrightarrow \frac{1}{2}\left[G(\omega-\omega_0) + G(\omega+\omega_0)\right]$$

This shows that the multiplication of a signal $g(t)$ by a sinusoid of frequency ω_0 shifts the spectrum $G(\omega)$ by $\pm\omega_0$. Multiplication of a sinusoid $\cos\omega_0 t$ by $g(t)$ amounts to modulating the sinusoid amplitude. This type of modulation is known as <u>amplitude modulation</u>. The sinusoid $\cos\omega_0 t$ is called the <u>carrier</u>, the signal $g(t)$ is the <u>modulating signal</u>, and the signal $g(t)\cos\omega_0 t$ is the <u>modulated signal</u>.



* Bandpass signals:

If $g_c(t)$ and $g_s(t)$ are low-pass signals, each with a bandwidth B Hz, then the signals $g_c(t)\cos\omega_0 t$ and $g_s(t)\sin\omega_0 t$ are both bandpass signals occupying the same band, each having a bandwidth of $4\pi B$ rad/s.
Hence, a general bandpass signal can be expressed as:
$$g_{bp}(t) = g_c(t)\cos\omega_0 t + g_s(t)\sin\omega_0 t$$

<u>3.6 Convolution:</u>

It is defined by the integral: $g(t)*w(t) = \int_{-\infty}^{\infty} g(\tau)\,w(t-\tau)\,d\tau$

* Time convolution property: $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$
$$\therefore\ g_1(t) * g_2(t) \Longleftrightarrow G_1(\omega)\,G_2(\omega)$$

* Frequency convolution property: $g_1(t)\,g_2(t) \Longleftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$
* Bandwidth of the product of 2 signals: $g_1(t)\,[B_1],\ g_2(t)\,[B_2] \Rightarrow g_1(t)g_2(t)\,[B_1+B_2]$

3.7 Time differentiation and time integration:

If $\quad g(t) \Leftrightarrow G(\omega)$

then (time differentiation): $\quad \frac{dg}{dt} \Leftrightarrow j\omega G(\omega)$

and (time integration): $\quad \int_{-\infty}^{t} g(\tau)\,d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\,\delta(\omega)$

Proof: * $\frac{dg}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, G(\omega)\, e^{j\omega t}\, d\omega$, $\quad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$

This shows that $\quad \frac{dg}{dt} \Leftrightarrow j\omega G(\omega)$

Repeated application of this property yields:

$$\frac{d^n g}{dt^n} \Leftrightarrow (j\omega)^n\, G(\omega).$$

* To prove the time integration one has to resort to:

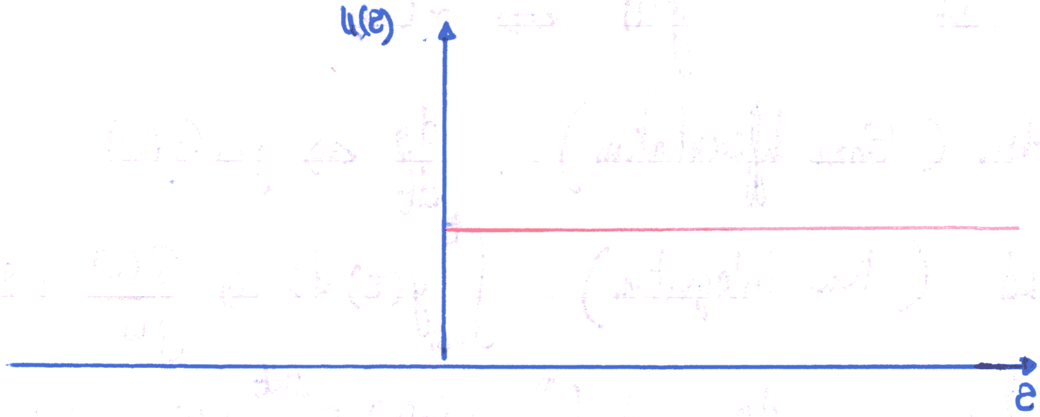
$$u(t-\tau) = \begin{cases} 1, & \tau \le t \\ 0, & \tau > t \end{cases}$$

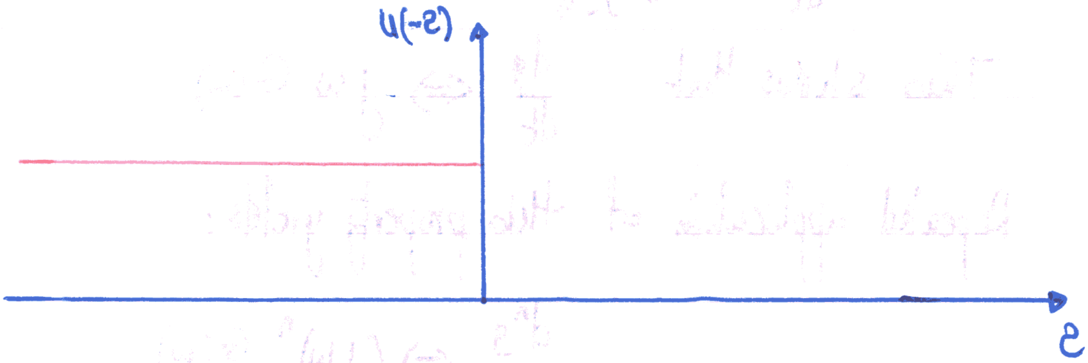
it follows that:

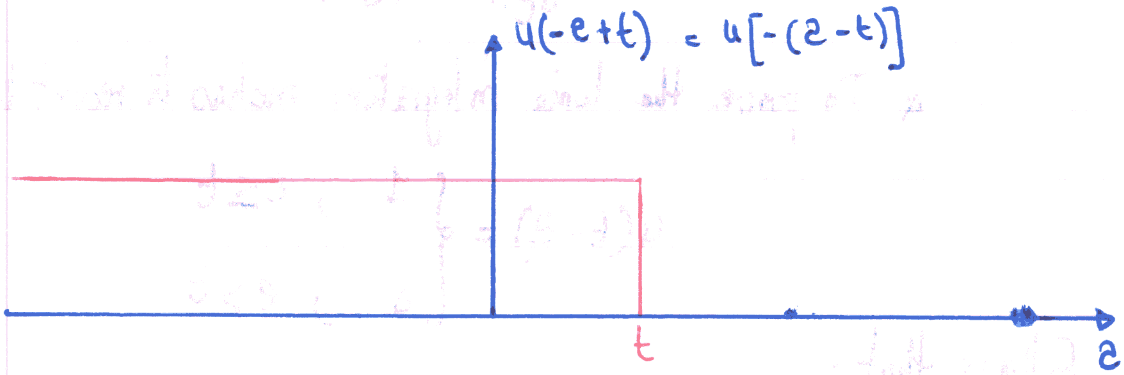
$$g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} g(\tau)\, d\tau$$

Now from the time convolution property, it follows that







$$g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} g(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} g(\tau) \times 1\, d\tau + \int_{t}^{+\infty} g(\tau) \times 0\, d\tau$$

$$= \int_{-\infty}^{t} g(\tau)\, d\tau$$

Table 3.1

Short Table of Fourier Transforms

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\ u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\ u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\ u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\ u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

$$g(t) * u(t) \Leftrightarrow G(\omega)\, U(\omega)$$
$$= G(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$
$$= \frac{G(\omega)}{j\omega} + \pi\, G(0)\,\delta(\omega)$$

4. Signal transmission through a linear system:

For a linear, time-invariant, continuous-time system the input-output relationship is given by:

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = g(t) * h(t)$$

where $g(t)$ is the input, $y(t)$ is the output, and $h(t)$ is the unit impulse response of the linear time-invariant system.

If $g(t) \Leftrightarrow G(\omega)$, $y(t) \Leftrightarrow Y(\omega)$, and $h(t) \Leftrightarrow H(\omega)$ [system transfer function]

then $\quad Y(\omega) = G(\omega)\, H(\omega)$

$G(\omega)$ and $Y(\omega)$ are the spectra of the input and the output, respectively.

This can be put as: (polar form)

$$|Y(\omega)|\, e^{j\theta_y(\omega)} = |G(\omega)|\, |H(\omega)|\, e^{j(\theta_g(\omega) + \theta_h(\omega))}$$

$$\therefore \quad |Y(\omega)| = |G(\omega)|\,|H(\omega)|$$
$$\theta_y(\omega) = \theta_g(\omega) + \theta_h(\omega)$$

$H(\omega)$ is called the frequency response of the system.

* Distortionless transmission:

In a distortionless transmission, the input $g(t)$ and the output $y(t)$ satisfy the condition: $\quad y(t) = k\, g(t - t_0)$

The Fourier transform of this equation yields

$$Y(\omega) = k\, G(\omega)\, e^{-j\omega t_0}$$

Since $Y(\omega) = G(\omega)\, H(\omega)$

$\Rightarrow \quad H(\omega) = k\, e^{-j\omega t_d}$

with $|H(\omega)| = k$ and $\theta_h(\omega) = -\omega t_d$

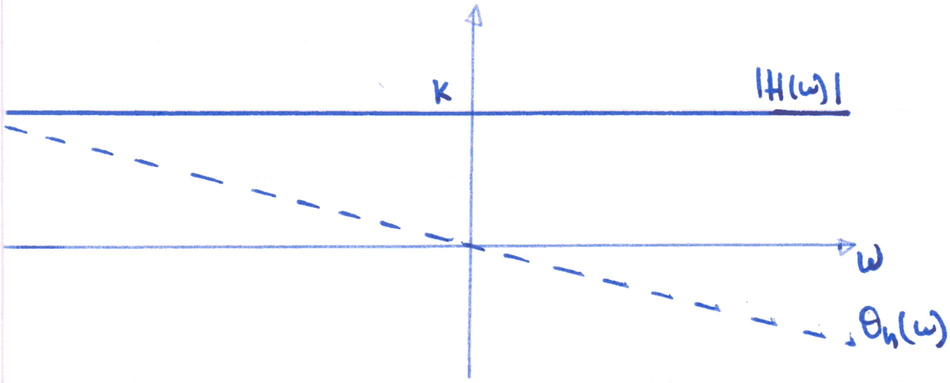


Fig. Linear T-I system frequency response for distortionless transmission.

[Note that the time delay resulting from the signal transmission through a system is given by: $\quad t_d(\omega) = -\dfrac{d\theta_h}{d\omega}$].

Therefore, for a distortionless system, t_d should be constant over the band of interest.

5. Ideal and practical Filters:

Ideal filters allow distortionless transmission of a certain band of frequencies and suppress all the remaining frequencies.

* Types of Filters:
  - Low-pass filters.
  - Band-pass filters.
  - High-pass filters.