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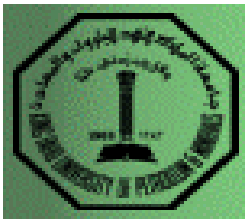

STUDY OF KALMAN FILTER

TERM # 012

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ABSTRACT

The Kalman filter is a mathematical power tool that is playing an increasingly important role in a number of applications like computer graphics, parametric estimation etc. The good thing with that one don't have to be a mathematical genius to understand and effectively use Kalman filters. While the Kalman filter has been around for about 30 years, it have recently started popping up in a wide variety of applications. The Kalman Filter is the best possible estimator for a large class of problems and a very effective and useful estimator for an even large class. Kalman filtering addresses an age-old question: How do you get accurate information out of inaccurate data? More pressingly, how do you update a "best" estimate for the state of a system as new, but still inaccurate, data pour in? Much as a coffee filter serves to keep undesirable grounds out of your morning mug, the Kalman filter is designed to strip unwanted noise out of a stream of data.

The applications of Kalman Filters are endless. Kalman filtering has proved useful in navigational and guidance systems, radar tracking, sonar ranging, and satellite orbit determination, to name just a few areas. Kalman and Bucy's original papers have generated thousands of other papers on aspects and applications of filtering. Their work has also stimulated mathematical research in such areas as numerical methods for linear algebra.

The *Extended Kalman Filter* (EKF) has become a standard technique used in a number of nonlinear estimation and machine learning applications. These include estimating the state of a nonlinear dynamic system, estimating parameters for nonlinear system identification (*e.g.*, learning the weights of a neural network), and dual estimation (*e.g.*, the Expectation Maximization (EM) algorithm) where both states and parameters are estimated simultaneously.

In this term project, I will provide a complete survey of Kalman Filter, its mathematical equations, and for better understanding I also apply Kalman filter to a nonlinear state estimation problem, in which we have to track a car moving on a circular track of fixed radius. Speed of the car is disturbed by the additive white gaussian noise, and we have to estimate its speed in the presence of gaussian noise, by using Extended Kalman Filter.

Chapter 1

Introduction

The term filter is often used to describe a device in the form of either hardware or computer software that is applied to a set of noisy data in order to extract information about a prescribed quantity of interest. The noise may arise from a variety of sources. For example, the data may have been derived by means of noisy sensors or may represent a useful signal component that has been corrupted by transmission through a communication channel. In any event, we may use the filter to perform basic operations like, filtering, smoothing and prediction. Here in this term project, we are basically more concerned with the application of filters for estimation problems.

A useful approach to filter-optimization problem is to minimize square value of the error signal that is defined as the different between some desired response and the actual filter output. For stationary inputs, the result is commonly known as Wiener Filter, which is also an optimum filter in mean square sense. The Wiener filter is inadequate for dealing with situations in which nonstationarity of the signal or noise is intrinsic to the problem. In such situations, the optimum filter has to be assume a time varying form. A highly successful solution to this more difficult problem is found in Kalman Filter.

1.1 Adaptive Filters

The design of a Wiener filter requires a priori information about the statistics of the data to be processed. The filter is optimum only when the statistical characteristics of the input data match the a priori information on which the design of the filter is based. When this information is not known completely, however, it may not be possible to design the Wiener filter or else the design may no longer be optimum. A very straight forward approach that is used in these situations is the “estimate” and plug procedure. This is a two stage process whereby the filter first “estimates” the statistical parameters of the relevant signals and than plug the results so obtained, into a nonrecursive formula for computing the filter parameters. For real time systems, this procedure has the

disadvantage of requiring excessively elaborate and costly hardware. A more efficient method is to use an adaptive filter. By such a device, we mean one that is self-designing in that the adaptive filter relies for its operation on a recursive algorithm, which makes it possible for the filter to perform satisfactorily in an environment where complete statistics of the relevant signal is not available. In a nonstationary environment, the algorithm offers a tracking capability, whereby it can track time variations in the statistics of the input data, provided that the variations are sufficiently slow.

1.2 Kalman Filter Approaches

There is no unique solution to the adaptive filtering problem. Rather, we have a kit of tools, represented by a variety of recursive algorithms, each of which offers desirable features of its own. The challenge facing the user of adaptive filtering is, first, to understand the capabilities and limitations of various adaptive filtering algorithms, and second to use this understanding in the selection of the appropriate algorithms for the application at hand.

Kalman filtering problem for a linear dynamic system is formulated in terms of two basic equations: the process equation that describe the dynamics of the system in terms of the state vector, and the measurement equation that describe measurement errors induced in the system. The solution to the problem is expressed as a set of time update recursions that are expressed in matrix form. To apply these recursions to solve the adaptive filtering problem, however, the theory requires that we postulate a model of the optimum operating conditions, which serve as a frame of reference for Kalman filter to track.

1.3 Basic Outline of the Report

This report basically provide a complete survey of Kalman Filter, its importance, its mathematical equations, and than a simple simulation for better understanding of the filter. One of the basic application of Kalman filters is state estimation, so I used Kalman filter for the estimation of a nonlinear state.

In chapter 2, I provides an introduction of probability theory and random variables, and I mainly focused on the behavior of random variables, and stochastic processes, because

the basics of Kalman filters lies in the theory of stochastic processes. Some of the fundamental definitions like probability, mean, variance etc. are discussed, and than I presents an brief overview of different distributions. Most commonly used distribution is the Gaussian distribution, which is very popular in modeling random systems for a variety of reasons, that's why my emphasis will mainly on random distributions

In chapter 3, a brief introduction of Kalman filter will be presented. One of the most fundamental application of Kalman filters is the process or state estimation. So here in this chapter, I will cover some of the fundamentals of process estimation, computational origin of the Kalman filter, how we can develop the equations of Kalman filters, what is the probabilistic origin of the filter. And than I switch to discrete Kalman filter algorithm, and the basic time and measurement update equations of the Discrete Kalman filter.

In chapter 4, I review the extended Kalman filter, which is used for the estimation of nonlinear states, which is the most likely case in most of the situations. Even here in this term project, I apply extended Kalman filter, because the process which I wants to be estimated is a nonlinear process. So here I modify the equations which I already derived in the previous section, for extended Kalman filters.

In chapter 5, I will briefly describe the simulation and my matlab code, which I wrote for the estimation of nonlinear state via extended Kalman filters. So here I briefly discuss the problem formulation, than how I apply Extended Kalman filter, and than what are the results, how Kalman filter track the actual state, these are the main topics of this chapter. I also describe my code, give the program listing, and show the results.

Chapter 2

Probability & Random Variable

What follows is a very basic introduction to probability and random variables. For more extensive coverage, see any book on the topic of probability theory and stochastic processes.

2.1 Probability

Most of us have some notion of what is meant by a "random" occurrence, or the probability that some event in a sample space will occur. Formally, the probability that the outcome of a discrete event (e.g. a coin flip) will favor a particular event is defined as:

$$p(A) = \frac{\text{Possible outcomes favoring event A}}{\text{Total number of possible outcomes}}$$

The probability of an outcome favoring either A or B is given by:

$$p(A \cup B) = p(A) + p(B)$$

If the probability of two outcomes is independent (one does not affect the other) than the, than the probability of both occurring is the product of their individual probabilities:

$$p(A \cap B) = p(A).p(B)$$

Finally, the probability of outcome A given an occurrence of outcome B is called the conditional probability of A given B, and is defined as

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$

2.2 Random Variables

As opposed to discrete events, in the motion of tracking and motion capture, we are more typically interested with the randomness associated with a continuous electrical voltage or perhaps a user's motion. In each case we can think of the item of interest as a continuous random variable. A random variable is essentially a function that maps all points in the sample space to real numbers. For example, the continuous random variable $X(t)$ might maps all points in the sample space to real numbers. For example, the continuous random variable $X(t)$ might map time to position. At any point in time, $X(t)$ would tell us the expected position.

In the case of continuous random variables, the probability of any single discrete event A is in fact 0. That is, $p(A) = 0$. Instead we can only evaluate the probability of events within some interval. A common function representing the probability of random variables is defined as *cumulative distribution function.*

$$F_X(x) = p(-\infty, x]$$

This function has some important properties defined as:

$$F_X(x) \to 0 \text{ as } x \to -\infty$$

$$F_X(x) \to 1 \text{ as } x \to +\infty$$

$$F_X(x) \text{ is a non decreasing function of } x$$

Even more commonly used equation is its derivative, which is called *probability density function.*

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Like cumulative distribution function, the probability density function also have following properties:

$$f_X(x) \qquad \text{is a non-negative function}$$

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1$$

Finally, note that the probability over any interval $[a,b]$ is defined as

$$p_X[a,b] = \int_a^b f_X(x)dx$$

2.3 Mean and Variance

Most of us familiar with the notion of the average of a sequence of numbers. For some N samples of a discrete random variable X, the average or sample mean is given by

$$\overline{X} = \frac{X_1 + X_2 + ... + X_N}{N}$$

Because in tracking we are dealing with continuous signals (with an uncountable sample space) it is useful to think in terms of an infinite number of trials, and correspondingly the outcome we would expect to see if we sampled the random variable infinitely, each time seeing one of n possible outcomes $x_1 ... x_n$. In this case, the expected value of the discrete random variable could be approximated by averaging probability-weighted events:

$$\overline{X} = \frac{(p_1 N)x_1 + (p_2 N)x_2 + ... + (p_n N)x_N}{N}$$

In effect, out of N trials, we would expect to see $(p_1 N)$ occurrences of event x_1 etc. This notion of infinite trials (samples) leads to the conventional definition of expected value for discrete random variables

$$\text{Expected value of } X = E(x) = \sum_{i=1}^{n} p_i x_i$$

For n possible outcomes, and there corresponding probabilities. Similarly, for the continuous random variable the expected value id defined as

$$\text{Expected value of } X = E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx$$

Finally, we note that above equations for expected values can be applied for the functions of the random variable X as follows:

$$E(g(x)) = \sum_{i=1}^{n} p_i g(x_i)$$

and for continuous,

$$E(g(x)) = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

The expected value of a random variable is also called first statistical moment. Similarly the k^{th} moment of a continuous random variable is given by:

$$E(X^k) = \int_{-\infty}^{+\infty} x^k f_X(x) dx$$

But normally we are interested in the second moment, which is given by:

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x) dx$$

When we let $g(X) = X - E(X)$ and apply the equation of second moment, so we can get the variance about the mean. In other words,

$$\text{Variance } X = E[(X - E(X))^2]$$

$$X = E(X^2) - E(X)^2$$

Variance is a very useful statistical property for random signals, because if we knew the variance of a signal that was otherwise supposed to be "constant" around some value - the mean, the magnitude of the variance would give us a sense how much jitter or "noise" is in the signal.

The square root of the variance, known as the standard deviation, it is also a useful statistical unit of measure because while being always positive, it has (as opposed to the variance) the same units as the original signal. The standard deviation is given by

$$\text{Standard deviation of } X = \sigma_X = \sqrt{\text{Variance of X}}$$

2.4 Normal or Gaussian Distribution

A special probability distribution known as Normal or Gaussian distribution has historically been popular in modeling random systems for a variety of reasons. As it turns out, many random processes occurring in nature actually appear to be normally distributed, or very close. In fact, under some particular conditions, it can be proved that a sum of random variables with any distribution tends toward a normal distribution, a very famous statement of *Central Limit Theorem.*

Given a random process $X \sim N(\mu, \sigma^2)$, i.e. a continuous random process X that is normally distributed with mean μ and variance σ^2. The probability density function for X is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

for $-\infty$ to $+\infty$. Any linear function of a normally distributed random process (variable) is also a normally distributed random process. Any linear function of a normally distributed random process is also a normally distributed random process. Graphically, the normal distribution is what is likely to be familiar as the "bell-shaped" curve shown below in figure:

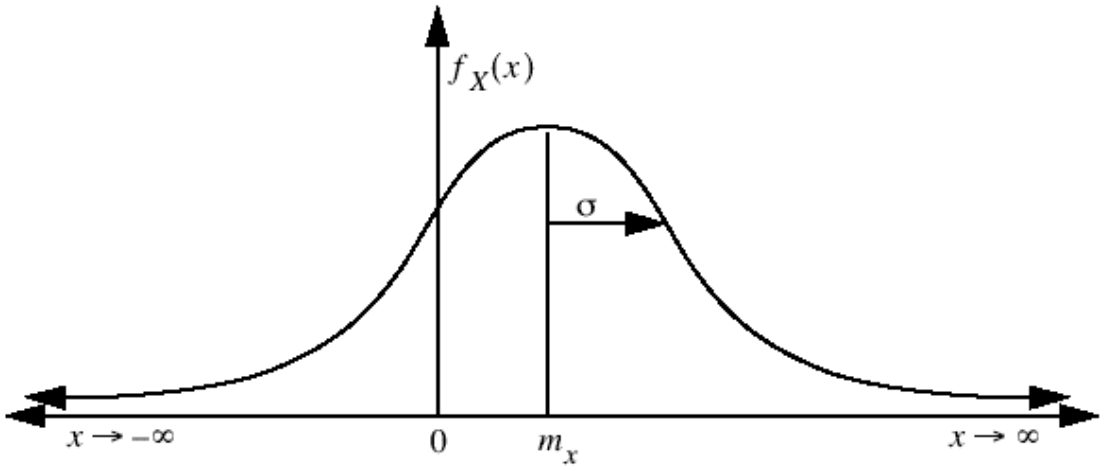


Figure2.1: The Normal or Gaussian probability distribution function

3.5 Continuous Independence and Conditional Probability

Two continuous random variables X and Y are said to be statistically independent, if their joint probability $f_{XY}(x, y)$ is equal to the product of their individual probabilities. In other words, they are considered independent if

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Bayes' Rule

In addition, Bayes rule offering a way to specify the probability density of the random variable X given (in the presence of) random variable Y. Bayes rule is given as

$$f_{X/Y}(x) = \frac{f_{Y/X}(y) f_X(x)}{f_Y(y)}$$

Continuous-Discrete

Given a discrete process X and a continuous process Y, the discrete probability mass function for X conditioned on Y = y is given by

$$p_X(x/Y = y) = \frac{f_Y(y/X = x) p_X(x)}{\sum_z f_Y(y/X = z) p_X(z)}$$

Note that this formula provides a discrete probability based on the conditioning density, without any integration.

2.6 Spatial vs. Spectral Signal Characteristics

In the previous sections we looked only at the spatial characteristics of random signals. As stated earlier, the magnitude of the variance of a signal can give us a sense of how much jitter or "noise" in the signal. However, a signals variance says nothing about the spacing or the rate of jitter over time. Here, we briefly discuss the temporal and hence spectral characteristics of a random signal.

A useful time-related characteristics of a random signal is autocorrelation, its correlation with itself over time. Formally, the autocorrelation of a random signal $X(t)$ is defined as:

$$R_X(t_1, t_2) = E[X(t_1) X(t_2)]$$

for sample times t_1 and t_2. If the process is stationary, than the autocorrelation depends only on the difference $\tau = t_1 - t_2$. In this special case, the autocorrelation can be re-written as

$$R_X(\tau) = E[X(t) X(t+\tau)]$$

Clearly the autocorrelation is a function of time, which means that it has a spectral interpretation in the frequency domain also. Again for stationary process, there is an important temporal-spectral relationship known as Wiener-Khinchine relation:

$$S_X(jw) = F[R_X(\tau)]$$

Chapter 3

Discrete Kalman Filter

In 1960, R.E. Kalman published his famous paper describing a recursive solution to the discrete-data linear filtering problem [Kalman60]. Since that time, due in large part to advances in digital computing, the Kalman filter has been the subject of extensive research and application, particularly in the area of autonomous or assisted navigation. A very "friendly" introduction to the general idea of the Kalman filter can be found in Chapter 1 of [Maybeck79], while a more complete introductory discussion can be found in [Sorenson70], which also contains some interesting historical narrative. More extensive references include [Gelb74; Grewal93; Maybeck79; Lewis86; Brown92; Jacobs93].

3.1 The Process to be Estimated

The Kalman filter addresses the general problem of trying to estimate the state $x \in \Re^n$ of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_k = Ax_{k-1} + Bu_k + w_{k-1}, \quad (3.1)$$

with a measurement $z \in \Re^m$ that is

$$z_k = Hx_k + v_k. \quad (3.2)$$

The random variables w_k and v_k represent the process and measurement noise (respectively). They are assumed to be independent (of each other), white, and with normal probability distributions

$$p(w) \sim N(0, Q), \quad (3.3)$$

$$p(v) \sim N(0, R). \quad (3.4)$$

In practice, the process noise covariance Q and measurement noise covariance R matrices might change with each time step or measurement, however here we assume they are constant.

The $n \times n$ matrix A in the difference equation (3.1) relates the state at the previous time step $k-1$ to the state at the current step k, in the absence of either a driving function or process noise. Note that in practice A might change with each time step, but here we assume it is constant. The $n \times l$ matrix B relates the optional control input $u \in \Re^l$ to the state x. The $m \times n$ matrix H in the measurement equation (3.2) relates the state to the measurement Z_k. In practice H might change with each time step or measurement, but here we assume it is constant.

3.2 The Computational Origins of the Filter

We define $\hat{x}_k^- \in \Re^n$ (note the "super minus") to be our a priori state estimate at step k given knowledge of the process prior to step k, and $\hat{x}_k \in \Re^n$ to be our a posteriori state estimate at step k given measurement z_k. We can then define a priori and a posteriori estimate errors as

$$e_k^- \equiv x_k - \hat{x}_k^-, \text{ and}$$
$$e_k \equiv x_k - \hat{x}_k$$

The a priori estimate error covariance is then

$$P_k^- = E[e_k^- e_k^{-T}], \quad (3.5)$$

and the a posteriori estimate error covariance is

$$P_k = E[e_k e_k^T]. \quad (3.6)$$

In deriving the equations for the Kalman filter, we begin with the goal of finding an equation that computes an a posteriori state estimate $\hat{x}_k$ as a linear combination of an a priori estimate $\hat{x}_k^-$ and a weighted difference between an actual measurement z_k and a measurement prediction $H\hat{x}_k^-$ as shown below in (3.7).

$$\hat{x}_k = \hat{x}_k^- + K(z_k - H\hat{x}_k^-) \quad (3.7)$$

The difference $(z_k - H\hat{x}_k^-)$ in (3.7) is called the measurement innovation, or the residual. The residual reflects the discrepancy between the predicted measurement $H\hat{x}_k^-$ and the actual measurement z_k. A residual of zero means that the two are in complete agreement.

The $n \times m$ matrix K in (3.7) is chosen to be the gain or blending factor that minimizes the a posteriori error covariance (3.6). This minimization can be accomplished by first substituting (3.7) into the above definition for e_k, substituting that into (3.6), performing the indicated expectations, taking the derivative of the trace of the result with respect to K, setting that result equal to zero, and then solving for K. For more details see [Maybeck79; Brown92; Jacobs93]. One form of the resulting K that minimizes (3.6) is given by

$$K_k = P_k^- H^T (H P_k^- H^T + R)^{-1}$$

$$= \frac{P_k^- H^T}{H P_k^- H^T + R} \quad . \text{(3.8)}$$

Looking at (3.8) we see that as the measurement error covariance R approaches zero, the gain K weights the residual more heavily. Specifically,

$$\lim_{R \to 0} K_k = H^{-1}$$

On the other hand, as the a priori estimate error covariance P_k^- approaches zero, the gain K weights the residual less heavily. Specifically,

$$\lim_{P_k^- \to 0} K_k = 0$$

Another way of thinking about the weighting by K is that as the measurement error covariance R approaches zero, the actual measurement z_k is "trusted" more and more, while the predicted measurement $H\hat{x}_k^-$ is trusted less and less. On the other hand, as the a priori estimate error covariance P_k^- approaches zero the actual measurement z_k is trusted less and less, while the predicted measurement $H\hat{x}_k^-$ is trusted more and more.

3.3 The Probabilistic Origins of the Filter

The justification for (3.7) is rooted in the probability of the a priori estimate $\hat{x}_k^-$ conditioned on all prior measurements z_k (Bayes' rule). For now let it suffice to point out that the Kalman filter maintains the first two moments of the state distribution,

$$E[x_k] = \hat{x}_k^-$$
$$E[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T] = P_k^-$$

The a posteriori state estimate (3.7) reflects the mean (the first moment) of the state distribution-- it is normally distributed if the conditions of (3.3) and (3.4) are met. The a posteriori estimate error covariance (3.6) reflects the variance of the state distribution (the second non-central moment). In other words,

$$p(x_k | z_k) \sim N(E[x_k], E[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T])$$
$$= N(\hat{x}_k, P_k).$$

For more details on the probabilistic origins of the Kalman filter, see [Maybeck79; Brown92; Jacobs93].

3.4 The Discrete Kalman Filter Algorithm

We will begin this section with a broad overview, covering the "high-level" operation of one form of the discrete Kalman filter (see the previous footnote). After presenting this high-level view, we will narrow the focus to the specific equations and their use in this version of the filter.

The Kalman filter estimates a process by using a form of feedback control: the filter estimates the process state at some time and then obtains feedback in the form of (noisy) measurements. As such, the equations for the Kalman filter fall into two groups: time update equations and measurement update equations. The time update equations are responsible for projecting forward (in time) the current state and error covariance estimates to obtain the a priori estimates for the next time step. The measurement update equations are responsible for the feedback--i.e. for incorporating a new measurement into the a priori estimate to obtain an improved a posteriori estimate.

The time update equations can also be thought of as predictor equations, while the measurement update equations can be thought of as corrector equations. Indeed the final estimation algorithm resembles that of a predictor-corrector algorithm for solving numerical problems as shown below in Figure 3-1.

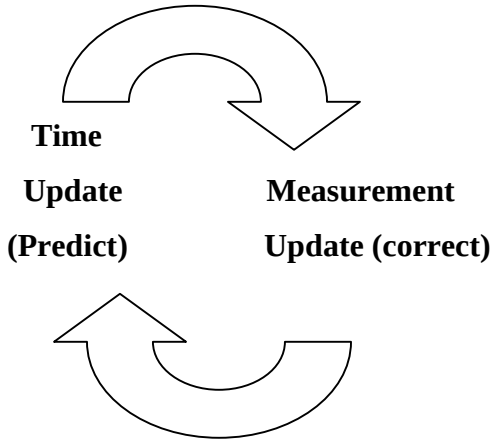


Figure 3-1. The ongoing discrete Kalman filter cycle. The time update projects the current state estimate ahead in time. The measurement update adjusts the projected estimate by an actual measurement at that time.

The specific equations for the time and measurement updates are presented below in Table 3-1 and Table 3-2.

Table 3-1: Discrete Kalman filter time update equations.

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k \quad (3.9)$$

$$P_k^- = AP_{k-1}A^T + Q \quad (3.10)$$

Again notice how the time update equations in Table 3-1 project the state and covariance estimates forward from time step $k-1$ to step k. A and B are from (3.1), while Q is from (3.3). Initial conditions for the filter are discussed in the earlier references.

Table 3-2: Discrete Kalman filter measurement update equations.

$$K_k = P_k^- H^T (H P_k^- H^T + R)^{-1} \quad (3.11)$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-) \quad (3.12)$$

$$P_k = (I - K_k H)P_k^- \quad (3.13)$$

The first task during the measurement update is to compute the Kalman gain, K_k. Notice that the equation given here as (3.11) is the same as (3.8). The next step is to actually measure the process to obtain z_k, and then to generate an a posteriori state estimate by incorporating the measurement as in (3.12). Again (3.12) is simply (3.7) repeated here for completeness. The final step is to obtain an a posteriori error covariance estimate via (3.13).

After each time and measurement update pair, the process is repeated with the previous a posteriori estimates used to project or predict the new a priori estimates. This recursive nature is one of the very appealing features of the Kalman filter--it makes practical implementations much more feasible than (for example) an implementation of a Wiener filter [Brown92] which is designed to operate on all of the data directly for each estimate. The Kalman filter instead recursively conditions the current estimate on all of the past measurements.Figure 3-2 below offers a complete picture of the operation of the filter, combining the high-level diagram of Figure 3-1 with the equations from Table 3-1 and Table 3-2.

3.5 Filter Parameters and Tuning

In the actual implementation of the filter, the measurement noise covariance R is usually measured prior to operation of the filter. Measuring the measurement error covariance R is generally practical (possible) because we need to be able to measure the process anyway (while operating the filter) so we should generally be able to take some off-line sample measurements in order to determine the variance of the measurement noise.

The determination of the process noise covariance Q is generally more difficult as we typically do not have the ability to directly observe the process we are estimating. Sometimes a relatively simple (poor) process model can produce acceptable results if one "injects" enough uncertainty into the process via the selection of Q. Certainly in this case one would hope that the process measurements are reliable.

In either case, whether or not we have a rational basis for choosing the parameters, often times superior filter performance (statistically speaking) can be obtained by tuning the filter parameters Q and R. The tuning is usually performed off-line, frequently with the help of another (distinct) Kalman filter in a process generally referred to as system identification.

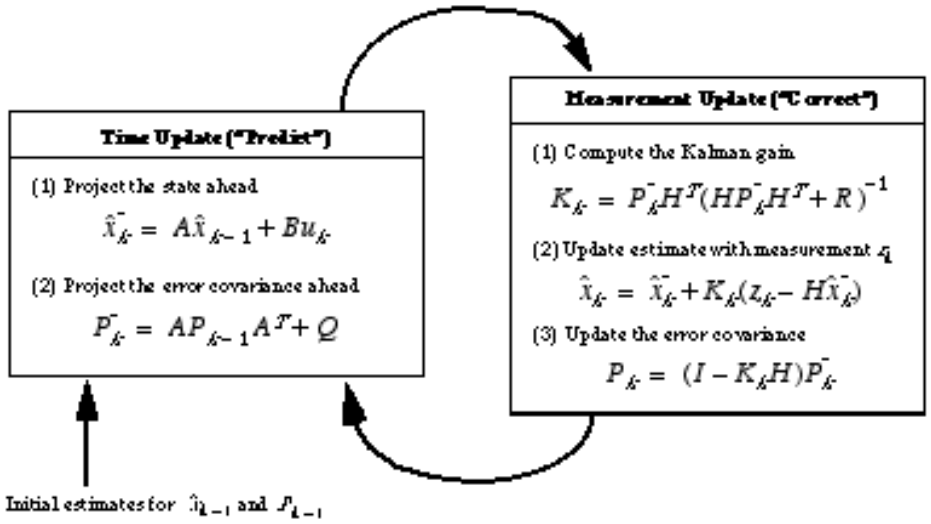


Figure 3-2. A complete picture of the operation of the Kalman filter, combining the high-level diagram of Figure 3-1 with the equations from Table 3-1 and Table 3-2

In closing we note that under conditions where Q and R are in fact constant, both the estimation error covariance P_k and the Kalman gain K_k will stabilize quickly and then remain constant (see the filter update equations in Figure 3-2). If this is the case, these parameters can be pre-computed by either running the filter off-line, or for example by determining the steady-state value of P_k as described in [Grewal93].

It is frequently the case however that the measurement error (in particular) does not remain constant. For example, when sighting beacons in our optoelectronic tracker

ceiling panels, the noise in measurements of nearby beacons will be smaller than that in far-away beacons. Also, the process noise Q is sometimes changed dynamically during filter operation becoming Q_k in order to adjust to different dynamics. For example, in the case of tracking the head of a user of a 3D virtual environment we might reduce the magnitude of Q_k if the user seems to be moving slowly, and increase the magnitude if the dynamics start changing rapidly. In such cases Q_k might be chosen to account for both uncertainty about the user's intentions and uncertainty in the model.

Chapter 4

Extended Kalman Filter

4.1 The Process to be Estimated

As described in previous chapter, the Kalman filter addresses the general problem of trying to estimate the state $x \in \Re^n$ of a discrete-time controlled process that is governed by a linear stochastic difference equation. But what happens if the process to be estimated and (or) the measurement relationship to the process is non-linear? Some of the most interesting and successful applications of Kalman filtering have been such situations. A Kalman filter that linearizes about the current mean and covariance is referred to as an extended Kalman filter or EKF.

In something akin to a Taylor series, we can linearize the estimation around the current estimate using the partial derivatives of the process and measurement functions to compute estimates even in the face of non-linear relationships. To do so, we must begin by modifying some of the material presented in previous chapter. Let us assume that our process again has a state vector $x \in \Re^n$, but that the process is now governed by the non-linear stochastic difference equation

$$x_k = f(x_{k-1}, u_k, w_{k-1}), \quad (4.1)$$

with a measurement $z \in \Re^m$ that is

$$z_k = h(x_k, v_k), \quad (4.2)$$

where the random variables w_k and v_k again represent the process and measurement noise as in (4.3) and (4.4). In this case the non-linear function f in the difference equation (4.1) relates the state at the previous time step $k-1$ to the state at the current time step k. It includes as parameters any driving function u_k and the zero-mean process noise w_k. The

non-linear function h in the measurement equation (4.2) relates the state x_k to the measurement z_k.

In practice of course one does not know the individual values of the noise w_k and v_k at each time step. However, one can approximate the state and measurement vector without them as

$$\bar{x}_k = f(\hat{x}_{k-1}, u_k, 0) \quad (4.3)$$

and

$$\bar{z}_k = h(\bar{x}_k, 0), \quad (4.4)$$

where $\hat{x}_k$ is some a posteriori estimate of the state (from a previous time step k).

It is important to note that a fundamental flaw of the EKF is that the distributions (or densities in the continuous case) of the various random variables are no longer normal after undergoing their respective nonlinear transformations. The EKF is simply an ad hoc state estimator that only approximates the optimality of Bayes' rule by linearization. Some interesting work has been done by Julier et al. in developing a variation to the EKF, using methods that preserve the normal distributions throughout the non-linear transformations [Julier96].

4.2 The Computational Origins of the Filter

To estimate a process with non-linear difference and measurement relationships, we begin by writing new governing equations that linearize an estimate about (4.3) and (4.4),

$$x_k = \bar{x}_k + A(x_{k-1} - \hat{x}_{k-1}) + Ww_{k-1}, \quad (4.5)$$

$$z_k = \bar{z}_k + H(x_k - \bar{x}_k) + Vv_k. \quad (4.6)$$

where

x_k and z_k are the actual state and measurement vectors,

$\bar{x}_k$ and $\bar{z}_k$ are the approximate state and measurement vectors from (4.3) and (4.4),

$\hat{x}_k$ is an a posteriori estimate of the state at step k,

the random variables w_k and v_k represent the process and measurement noise as in (4.3) and (4.4).

A is the Jacobian matrix of partial derivatives of f with respect to x, that is

$$A_{[i,j]} = \frac{\partial f_{[i]}}{\partial x_{[j]}}(\hat{x}_{k-1}, u_k, 0),$$

W is the Jacobian matrix of partial derivatives of f with respect to w,

$$W_{[i,j]} = \frac{\partial f_{[i]}}{\partial w_{[j]}}(\hat{x}_{k-1}, u_k, 0),$$

H is the Jacobian matrix of partial derivatives of h with respect to x,

$$H_{[i,j]} = \frac{\partial h_{[i]}}{\partial x_{[j]}}(\bar{x}_k, 0),$$

V is the Jacobian matrix of partial derivatives of h with respect to v,

$$V_{[i,j]} = \frac{\partial h_{[i]}}{\partial v_{[j]}}(\bar{x}_k, 0).$$

Note that for simplicity in the notation we do not use the time step subscript k with the Jacobians A, W, H, and V, even though they are in fact different at each time step.

Now we define a new notation for the prediction error,

$$\bar{e}_{x_k} \equiv x_k - \bar{x}_k, \quad (4.7)$$

and the measurement residual,

$$\bar{e}_{z_k} \equiv z_k - \bar{z}_k. \quad (4.8)$$

Remember that in practice one does not have access to x_k in (4.7), it is the actual state vector, i.e. the quantity one is trying to estimate. On the other hand, one does have access to z_k in (4.8), it is the actual measurement that one is using to estimate x_k. Using (4.7) and (4.8) we can write governing equations for an error process as

$$\bar{e}_{x_k} = A(x_{k-1} - \hat{x}_{k-1}) + \varepsilon_k, \quad (4.9)$$

$$\bar{e}_{z_k} = H\bar{e}_{x_k} + \eta_k, \quad (4.10)$$

where ε_k and η_k represent new independent random variables having zero mean and covariance matrices WQW^T and VRV^T, with Q and R as in (4.3) and (4.4) respectively.

Notice that the equations (4.9) and (4.10) are linear, and that they closely resemble the difference and measurement equations (4.1) and (4.2) from the discrete Kalman filter.

This motivates us to use the actual measurement residual $\tilde{e}_{z_k}$ in (4.8) and a second (hypothetical) Kalman filter to estimate the prediction error $\tilde{e}_{x_k}$ given by (4.9). This estimate, call it $\hat{e}_k$, could then be used along with (4.7) to obtain the a posteriori state estimates for the original non-linear process as

$$\hat{x}_k = \bar{x}_k + \hat{e}_k. \quad (4.11)$$

The random variables of (4.9) and (4.10) have approximately the following probability distributions (see the previous footnote):

$$p(\tilde{e}_{x_k}) \sim N(0, E[\tilde{e}_{x_k}\tilde{e}_{x_k}^T])$$
$$p(\varepsilon_k) \sim N(0, WQ_kW^T)$$
$$p(\eta_k) \sim N(0, VR_kV^T)$$

Given these approximations and letting the predicted value of $\hat{e}_k$ be zero, the Kalman filter equation used to estimate $\hat{e}_k$ is

$$\hat{e}_k = K_k\tilde{e}_{z_k}. \quad (4.12)$$

By substituting (4.12) back into (4.11) and making use of (4.8) we see that we do not actually need the second (hypothetical) Kalman filter:

$$\begin{aligned}\hat{x}_k &= \bar{x}_k + K_k\tilde{e}_{z_k} \\ &= \bar{x}_k + K_k(z_k - \bar{z}_k)\end{aligned} \quad (4.13)$$

Equation (4.13) can now be used for the measurement update in the extended Kalman filter, with $\bar{x}_k$ and $\bar{z}_k$ coming from (4.3) and (4.4), and the Kalman gain K_k coming from (4.11) with the appropriate substitution for the measurement error covariance.

The complete set of EKF equations is shown below in Table 4-1 and Table 4-2. Note that we have substituted $\hat{x}_k^-$ for $\bar{x}_k$ to remain consistent with the earlier "super minus" a priori notation, and that we now attach the subscript k to the Jacobians A, W, H, and V, to reinforce the notion that they are different at (and therefore must be recomputed at) each time step.

Table 4-1: EKF time update equations.

$$\hat{x}_k^- = f(\hat{x}_{k-1}, u_k, 0) \quad (4.14)$$

$$P_k^- = A_k P_{k-1} A_k^T + W_k Q_{k-1} W_k^T \quad (4.15)$$

As with the basic discrete Kalman filter, the time update equations in Table 4-1 project the state and covariance estimates from the previous time step $k-1$ to the current time step k. Again f in (4.14) comes from (4.3), A_k and W_k are the process Jacobians at step k, and Q_k is the process noise covariance (4.3) at step k.

Table 4-2: EKF measurement update equations.

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + V_k R_k V_k^T)^{-1} \quad (4.16)$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h(\hat{x}_k^-, 0)) \quad (4.17)$$

$$P_k = (I - K_k H_k) P_k^- \quad (4.18)$$

As with the basic discrete Kalman filter, the measurement update equations in Table 4-2 correct the state and covariance estimates with the measurement z_k. Again h in (4.17) comes from (4.4), H_k and V are the measurement Jacobians at step k, and R_k is the measurement noise covariance (4.4) at step k. (Note we now subscript R allowing it to change with each measurement).

The basic operation of the EKF is the same as the linear discrete Kalman filter as shown in Figure 4-1. Figure 4-1 below offers a complete picture of the operation of the EKF, combining the high-level diagram of Figure 4-1 with the equations from Table 5-4 and Table 4-2.

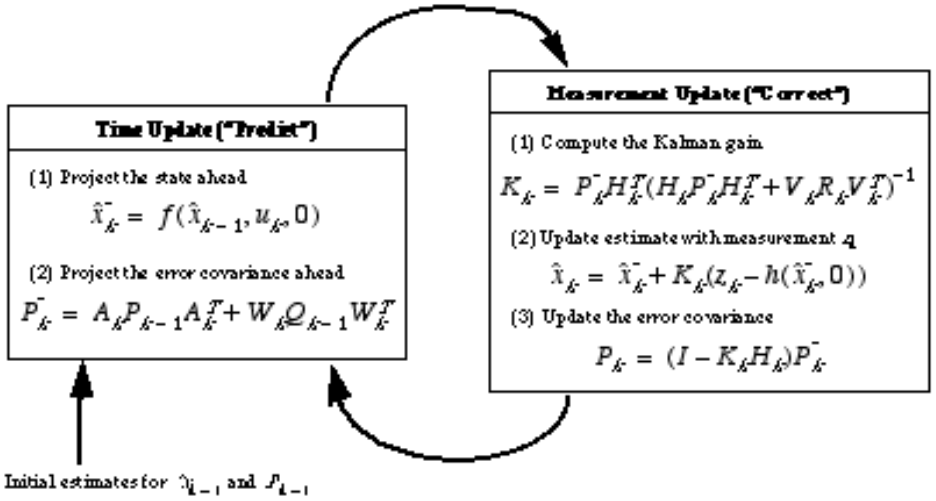


Figure 4-1. A complete picture of the operation of the extended Kalman filter, combining the high-level diagram of Figure 3-1 with the equations from Table 4-1 and Table 4-2.

An important feature of the EKF is that the Jacobian H_k in the equation for the Kalman gain K_k serves to correctly propagate or "magnify" only the relevant component of the measurement information. For example, if there is not a one-to-one mapping between the measurement z_k and the state via h, the Jacobian H_k affects the Kalman gain so that it only magnifies the portion of the residual $z_k - h(\hat{x}_k^-, 0)$ that does affect the state. Of course if over all measurements there is not a one-to-one mapping between the measurement z_k and the state via h, then as you might expect the filter will quickly diverge. In this case the process is unobservable.

4.3 Applications

The EKF has been applied extensively to the field of nonlinear estimation. General application areas may be divided into *state-estimation* and *machine learning*. We further divide machine learning into *parameter estimation* and *dual estimation*. The framework for these areas are briefly reviewed next.

4.3.1 State Estimation

The basic framework for the EKF involves estimation of the state of a discrete-time nonlinear dynamic system,

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k, \mathbf{v}_k) \quad (1)$$

$$\mathbf{y}_k = H(\mathbf{x}_k, \mathbf{n}_k), \quad (2)$$

where $\mathbf{x}_k$ represent the unobserved state of the system and $\mathbf{y}_k$ is the only observed signal. The *process* noise $\mathbf{v}_k$ drives the dynamic system, and the *observation* noise is given by $\mathbf{n}_k$. Note that we are not assuming additivity of the noise sources. The system dynamic model F and H are assumed known.

In state-estimation, the EKF is the standard method of choice to achieve a recursive (approximate) maximum-likelihood estimation of the state $\mathbf{x}_k$.

4.3.2 Parameter Estimation

The classic machine learning problem involves determining a nonlinear mapping

$$\mathbf{y}_k = G(\mathbf{x}_k, \mathbf{w}) \quad (3)$$

where $\mathbf{x}_k$ is the input, $\mathbf{y}_k$ is the output, and the nonlinear map G is parameterized by the vector $\mathbf{w}$. The nonlinear map, for example, may be a feedforward or recurrent neural network ($\mathbf{w}$ are the weights), with numerous applications in regression, classification, and dynamic modeling. Learning corresponds to estimating the parameters $\mathbf{w}$. Typically, a training set is provided with sample pairs consisting of known input and desired outputs, $\{\mathbf{x}_k, \mathbf{d}_k\}$. The error of the machine is defined as $\mathbf{e}_k = \mathbf{d}_k - G(\mathbf{x}_k, \mathbf{w})$, and the goal of learning involves solving for the parameters $\mathbf{w}$ in order to minimize the expected squared error.

While a number of optimization approaches exist (*e.g.*, gradient descent using backpropagation), the EKF may be used to estimate the parameters by writing a new state-space representation

$$\mathbf{w}_k = \mathbf{w}_{k-1} + \mathbf{u}_k \quad (4)$$

$$\mathbf{y}_k = G(\mathbf{x}_k, \mathbf{w}_k) + e_k. \quad (5)$$

where the parameters $\mathbf{w}_k$ correspond to a stationary process with identity state transition matrix, driven by process noise $\mathbf{u}_k$ (the choice of variance determines tracking performance). The output $\mathbf{y}_k$ corresponds to a nonlinear observation on $\mathbf{w}_k$. The EKF can then be applied directly as an efficient ``second-order" technique for learning the parameters. In the linear case, the relationship between the Kalman Filter (KF) and Recursive Least Squares (RLS) is given in [2]. The use of the EKF for training neural networks has been developed by Singhal and Wu [3] and Puskorious and Feldkamp [4].

4.3.3 Dual Estimation

A special case of machine learning arises when the input $\mathbf{x}_k$ is unobserved, and requires coupling both state-estimation and parameter estimation. For these *dual estimation* problems, we again consider a discrete-time nonlinear dynamic system,

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k, \mathbf{v}_k, \mathbf{w}) \quad (6)$$

$$\mathbf{y}_k = H(\mathbf{x}_k, \mathbf{n}_k, \mathbf{w}). \quad (7)$$

where both the system states $\mathbf{x}_k$ and the set of model parameters $\mathbf{w}$ for the dynamic system must be simultaneously estimated from only the observed noisy signal $\mathbf{y}_k$.
In the next section we explain the basic assumptions and flaws with the using the EKF. In Section 6.1, we introduce the Unscented Kalman Filter (UKF) as a method to amend the flaws in the EKF.

4.4 Basic Flaw In EKF

Consider the basic state-space estimation framework as in Equations 1 and 2. Given the noisy observation $\mathbf{y_k}$, a recursive estimation for $\mathbf{x_k}$ can be expressed in the form (see [5]),

$$\mathbf{x}_k) + K_k \cdot [\mathbf{y}_k - (\text{prediction of } \mathbf{y}_k)] \quad (8)$$

This recursion provides the optimal minimum mean-squared error (MMSE) estimate for $\mathbf{x}_k$ assuming the prior estimate $\hat{\mathbf{x}}_{k-1}$ and current observation $\mathbf{y}_k$ are Gaussian Random Variables (GRV). We need not assume linearity of the model. The optimal terms in this recursion are given by

$$\hat{\mathbf{x}}_k^- = E[F(\hat{\mathbf{x}}_{k-1}, \mathbf{v}_{k-1})] \quad (9)$$

$$K_k = \mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} \mathbf{P}_{\hat{\mathbf{y}}_k \hat{\mathbf{y}}_k}^{-1} \quad (10)$$

$$\hat{\mathbf{y}}_k^- = E[H(\hat{\mathbf{x}}_k^-, \mathbf{n}_k)], \quad (11)$$

where the optimal prediction of $\mathbf{x}_k$ is written as $\hat{\mathbf{x}}_k^-$, and corresponds to the expectation of a nonlinear function of the random variables $\hat{\mathbf{x}}_{k-1}$ and $\mathbf{v}_{k-1}$ (similar interpretation for the optimal prediction $\hat{\mathbf{y}}_k^-$). The optimal gain term K_k is expressed as a function of posterior covariance matrices (with $\hat{\mathbf{y}}_k = \mathbf{y}_k - \hat{\mathbf{y}}_k^-$). Note these terms also require taking expectations of a nonlinear function of the prior state estimates.

The Kalman filter calculates these quantities exactly in the linear case, and can be viewed as an efficient method for analytically propagating a GRV through linear system dynamics. For nonlinear models, however, the EKF *approximates* the optimal terms as:

$$\hat{\mathbf{x}}_k^- \approx F(\hat{\mathbf{x}}_{k-1}, \bar{\mathbf{v}}) \quad (12)$$

$$K_k \approx \hat{\mathbf{P}}_{\mathbf{x}_k \mathbf{y}_k} \hat{\mathbf{P}}_{\tilde{\mathbf{y}}_k \tilde{\mathbf{y}}_k}^{-1} \quad (13)$$

$$\hat{\mathbf{y}}_k^- \approx H(\hat{\mathbf{x}}_k^-, \bar{\mathbf{n}}), \quad (14)$$

where predictions are approximated as simply the function of the prior *mean* value for estimates (no expectation taken) The covariance are determined by linearizing the dynamic equations $(\mathbf{x}_{k+1} \approx \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{v}_k, \mathbf{y}_k \approx \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{n}_k)$, and then determining the posterior covariance matrices analytically for the linear system. In other words, in the EKF the state distribution is approximated by a GRV which is then propagated analytically through the ``first-order" linearization of the nonlinear system. The readers are referred to [5] for the explicit equations. As such, the EKF can be viewed as providing ``first-order" approximations to the optimal terms. These approximations, however, can introduce large errors in the true posterior mean and covariance of the transformed (Gaussian) random variable, which may lead to sub-optimal performance and sometimes divergence of the filter. It is these ``flaws" which will be amended in the next section using the UKF.

Chapter 5

Simulations

In this term project, I basically try to overview the basic characterstics of kalman filter, its mathematical equations, and demonstate how simple is, to use the kalman filter. Beacuase the beauty of kalman filter is that, in order to use it, you don't need to know much more about its mathematical complexity, with a very superficial knowledge, one can apply Kalman Filter directly to any application easily.

So, here in order to understand Kalman Filter more clearly I apply Kalman Filter to a nonlinear state estimation problem, in which, EKF is used to track a car which is moving on a circular arc of fixed radius, which I assumed here 10. The speed of the car is disturbed by the white gausian noise along the arc. So by using EKF, I try to estimate the speed of the car, along the curve. The only observation which the filters have, is the bearing angle of car.

5.1 Tracking the Car Along a Arc of Fixed Radius

Now, in this program, program first asked about the number of runns, from the user, means how many times one wants to repeat the simulation. You can repeat the tracking of filters, as many times as you want, and each time u expect different result, because the noise is random in behaviour. Now here in this program first of all, I initialized certain variables by their default values, these variables includes covariance (process and measurement), radius, which we can choose any arbitraily value, here I assure radius of the circulat track is 10, speed of the car is assumed to be unity, number of time stpes are assumed to be 100 and so on. Now after initialize all the variables, generate the measurement and process noise by using "randn" function of the matlab, which will generate normaly distributed random gata. Now for initail observation of states, I have to call the function "hfun1" by using matlab function "feval", and store the initial states of the system. Now estimate the state by applying EKF, so start a loop, first of all predict the mean, than calculate jacobian matix for calculating the covariance, and than predict the state of the system by utilizing the equations already described in chapter # 4. After

calculating all the estimated values, than calculating the error, simply by taking the difference between the actual value and the predicted value of state. Now than taking its square root, and calculating the mean square error of prediction. Ideally this error should be zero. Finally I will display all the results as shown below.

5.2 Source Code

```
clc;            % Clear work space
close all;      % Close all windows
clear all;      % Clear all variables

number_of_runs = input('Number of runs : ');
                                % Number of times program run

mean_RMSE_ekf = zeros(1,number_of_runs);
                                % Mean root mean square error

for j=1:number_of_runs,         % Main loop begins

radius = 10;    % Set arbitraily fixed value
speed  = 1;             % Speed is assumed to be unity
dt     = 1;             % Dummy variable

u = [radius; dt];       % Control inputs

N = 100;                % Number of time steps

x0  = [speed; radius; 0];  % Initial state

P0  = 1*[1 0 0;
         0 1 0;
         0 0 1];    % Initial state covariance

L = size(x0,1);         % State dimensions

Q    = (.1*speed)^2;    % Process noise variance
R    = 0.1;             % Measurement noise variance
```

```
xh  = zeros(L,N+1);  % State estimate buffer
P   = zeros(L,L,1,N+1);% State covariance buffer

xh(:,1)    = x0;       % initialize buffers
Px(:,:,1)  = P0;

xh_ekf = xh;          % Create EKF buffers from template
P_ekf  = Px;

alpha = 1;
beta  = 2;
kappa = 0;          % 3 - state dimension

%%-------------------------------------------------------------------
%%---------------------- GENERATE DATASET --------------------------

fprintf('\nGenerating data...\n');

x      = zeros(L,N+1);
y      = zeros(1,N+1);
v      = sqrt(Q)*randn(1,N+1);    % Process noise
n      = sqrt(R)*randn(1,N+1);    % Measurement noise

x(:,1) = x0;      % Initial state condition

y(:,1) = feval('hfun1',x(:,1),u,n(:,1),1);
                                        % Initial onbservation of state

for k=2:(N+1),
  x(:,k) = feval('ffun1',x(:,k-1),u,v(:,k),k);
```

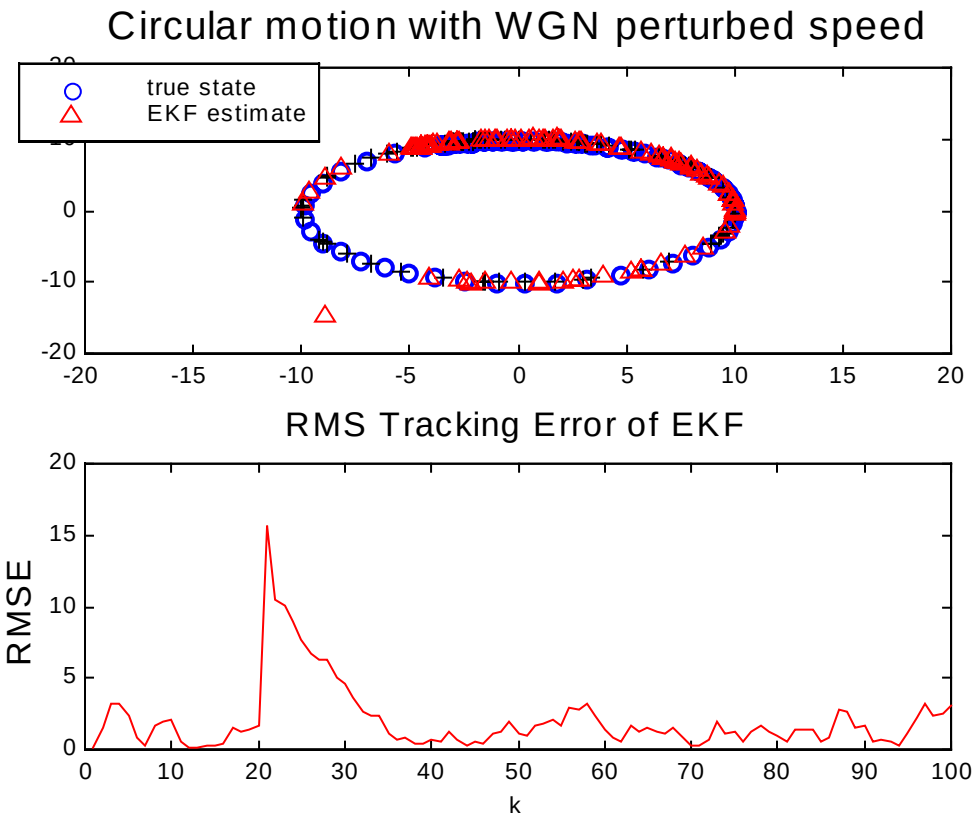
```
  y(:,k) = feval('hfun1',x(:,k),u,n(:,k),k);
end



%%--------------------------------------------------------------------
%%------------------- ESTIMATE STATE USING EKF ----------------------

fprintf('\nEstimating trajectory...\n');

for k=2:(N+1),

  % Generate EKF estimate
  xPred_ekf = feval('ffun1',xh_ekf(:,k-1),u,0,k);
                                          % EKF predicted mean
  Jx = jacobian_ffun1(xh_ekf(:,k-1),u);
              % Jacobian for ffun1
  PPred_ekf = diag([Q 0 0]) + Jx*P_ekf(:,:,k-1)*Jx';
                                          % EKF predicted state covariance
  yPred = feval('hfun1',xPred_ekf,u,0,k);
  Jy = jacobian_hfun1(xPred_ekf,u);  % Jacobian for hfun1
  % Calculations
  S  = R + Jy*PPred_ekf*Jy';
  Si = inv(S);
  K = PPred_ekf*Jy'*Si;
  xh_ekf(:,k)  = xPred_ekf + K*(y(:,k)-yPred);
                                          % Predicted state estimate
  P_ekf(:,:,k) = PPred_ekf - K*Jy*PPred_ekf;

end

%%--------------------------------------------------------------------
```

```
%%------------------- CALCULATE ERRORS ------- ----------------------

error_ekf = (x(:,2:end)-xh_ekf(:,2:end)).^2;
RMSE_ekf  = (sum(error_ekf).^0.5);
mean_RMSE_ekf(j) = mean(RMSE_ekf);



fprintf('\n\nEKF estimate normalized RMSE : %2.4f\n',mean_RMSE_ekf(j)/radius);

%%--------------------------------------------------------------------
%%------------------- DISPLAY RESULTS -----------------------------

figure(1); clf;
subplot(211);
p1 = plot(x(2,:),x(3,:),'bo','linewidth',1.5); hold on
p2 = plot(radius*cos(y),radius*sin(y),'k+');
p3 = plot(xh_ekf(2,:),xh_ekf(3,:),'r^');
legend([p1 p3],'true state','EKF estimate');
title('Circular motion with WGN perturbed speed','fontsize',16);
axis(2*[-radius radius -radius radius]);
subplot(212);
p1=plot(RMSE_ekf,'r'); hold on;
title('RMS Tracking Error of EKF','fontsize',14);
xlabel('k');
ylabel('RMSE','fontsize',14);
drawnow

end

fprintf('\n\n');
fprintf('-------------------------------------------------------\n');
```

```
fprintf('Mean & Variance of normalized RMSE over %d runs\n\n',number_of_runs);
fprintf('EKF                    :                    %2.4f
(%2.4f)\n',mean(mean_RMSE_ekf/radius),var(mean_RMSE_ekf/radius));
fprintf('-------------------------------------------------------\n');
```

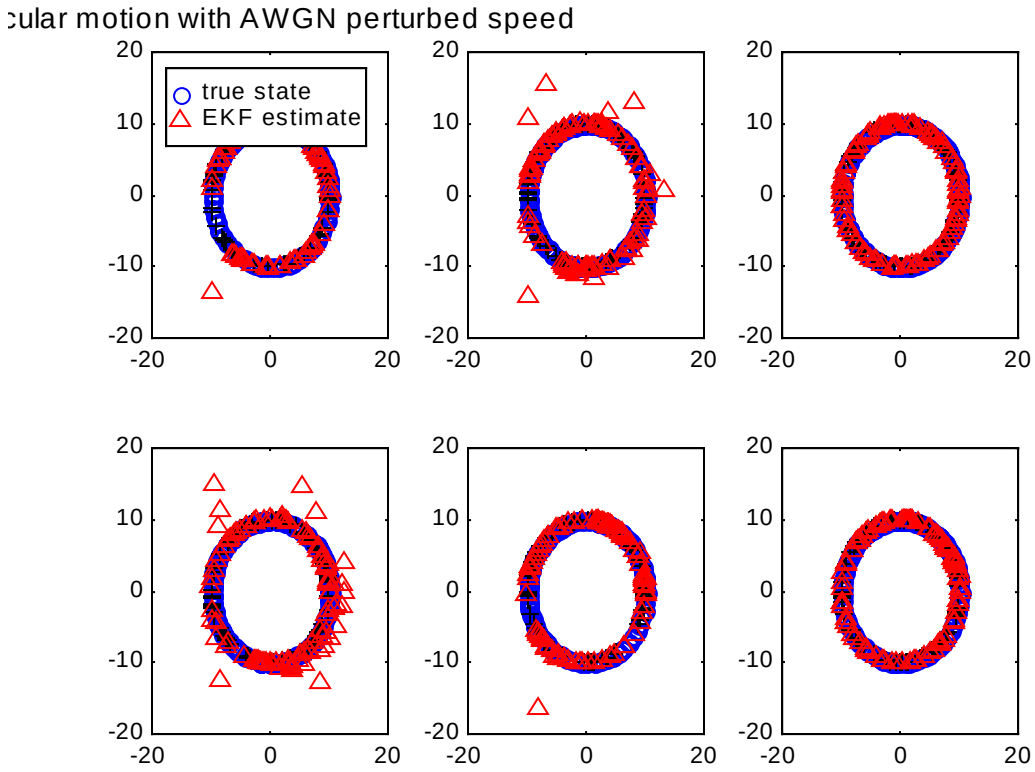
5.3 Simulation Results

Now when we run the above program, only once, and try to estimate the speed of the car which is moving on a circular arc of radius 10, we have following results, how the EKF track the actual state, as well as I also plot here the mean square error for the reference, because it shows clrealy how much the EKF succeded in estimating the state.

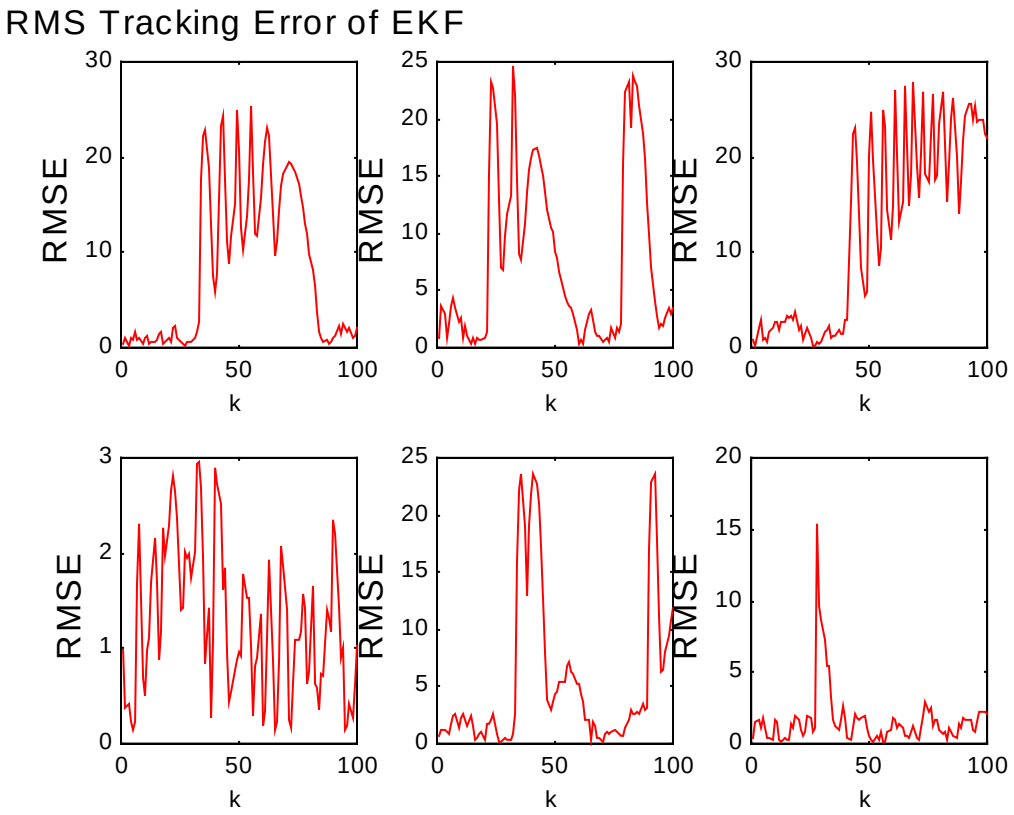


Similarly, for many runs, the screen will show you the trajectory traced by the EKF in each run. The filter, basically calculate the bearing angle in each case, and than based on that angle, they will estimate the current states, by using relations already described earlier.

So for many runs,



Similarlay, we can plot the RMS tracking error, in each case, as we already know, lower the tracking error better will be the performance. So, the RMS tracking error is given as,



So, its quite clear from the above curves of mean square error, that sometimes the mean square error is high and sometimes its settles to a moderate level.

5.2 Conclusion

Here in this term project, I review the basic concepts of estimation in the light of Kalman Filter. Kalman Filter is basically a set of mathematical equations which are used for the estimation of states. For nonlinear state estimation problems, we use extended kalman filter, which is a very powerful mathematical tool. After studing the basics of kalman filters, I than apply kalman filter on a nonlinear state estimation problem. Form the above simulation results, its quite clear that extended kalman filter can be successfully used for the nonlinear state estimation problem, since the process is random, so in each itteration we have different mean square error, but in general, the level of mean square error, is within permissible limits, so EKF can be used without any doubt to nonlinea estimation problem. Anyone can also extend my work, so there is also a suggestion for those who have interest in this area, and wants to extend my work. There is another modification of

EKF so called UKF (Uncented Kalman Filter) whose performance is better than EKF, so anybody can extend this work by applying UKF on the same problem, and than compare the results of the two filters.

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