Chapter 2

Dielectric Slab Optical Waveguide

2.1 Introduction

Dielectric slab waveguides are the simplest optical waveguiding structures. Because of their simple geometry, guided and radiation modes can be described by simple mathematical expressions. The study of slab waveguides is important in gaining understanding of the wave-guiding properties of more complicated dielectric waveguides. It must be noted, however, that slab waveguides are not only useful as models for more general types of optical waveguides but they are actually employed for light guidance in integrated optical circuits [39, 40, 41, 42, 43].

In this chapter, the theory of planar dielectric waveguides will be explained. Starting with Maxwell's equations, we will obtain the details of the propagation of optical modes in a slab waveguide.

2.2 Maxwell's Equations

The propagation of electromagnetic waves in dielectric media is governed by Maxwell's equations which are [40, 41]:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2.1}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{2.4}$$

where

E is the electric field strength.

H is the magnetic field strength.

B is the magnetic flux density.

D is the electric displacement.

J is the electric current density.

ρ is the electric charge density.

The basic four quantities **E**, **B**, **H** and **D** are vectors in the three-dimensional space. They are generally functions of both space and time.

2.3 Dielectric Slab Waveguide

Figure 2.1 shows a schematic diagram of a three-layer planar waveguide. The core region of the waveguide, which is also called the film, is assumed to have refractive index n_1. The film is deposited on a layer called substrate which has a refractive index n_2. The cladding on the film is called superstrate and it has a refractive index n_3.

The behavior of dielectric waveguides can be explained with the aid of the three-layer model shown in figure 2.2. As can be seen, this figure is a longitudinal cross-section of the slab waveguide shown in figure 2.1. In figure 2.2, we assume that the dimension of the slab along the y-axis is considerably larger than its dimension along the x-axis and that no material or field variation exist along the y-direction. Such a waveguide supports a finite number of guided modes as well as an infinite number of unguided radiation modes. In order to achieve mode guidance, it is necessary that n_1 be greater than n_2 and n_3 that is $n_1 > n_2 \geq n_3$. If $n_2 = n_3$, the slab waveguide is said to be symmetric. For the case of figure 2.2 where n_2 is different from n_3, the slab waveguide is asymmetric. For a symmetric waveguide, the guided modes are either even or odd in their field distributions, as shown in figure 2.3. This waveguide can be considered to constitute a limiting case of an asymmetric waveguide. As will be analytically shown in the subsequent sections of this chapter, the number of guided modes that can be supported by a slab waveguide depends on the thickness $2d$, the wavelength λ and the indices of refraction, n_1, n_2 and n_3.

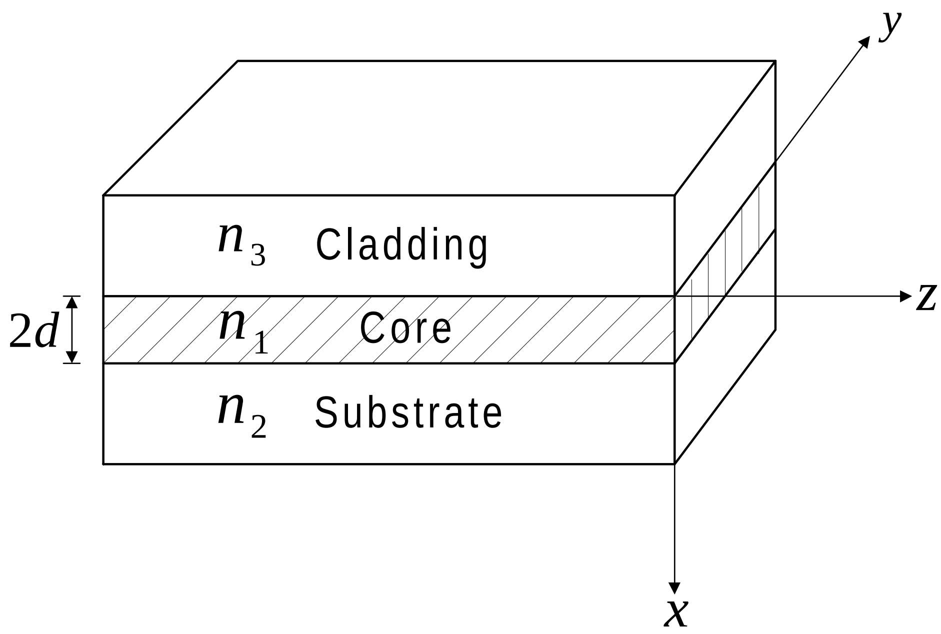

Figure 2.1: Schematic Diagram of A Dielectric Slab Waveguide

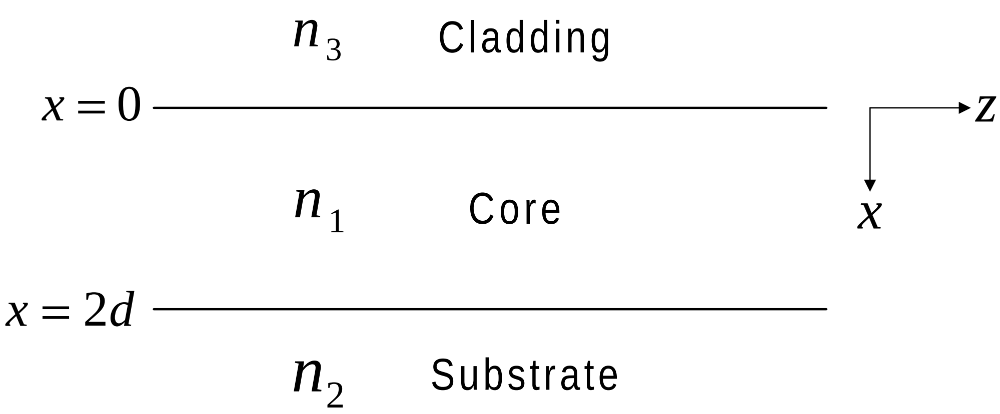

Figure 2.2: A Three-Layer Dielectric Slab Waveguide

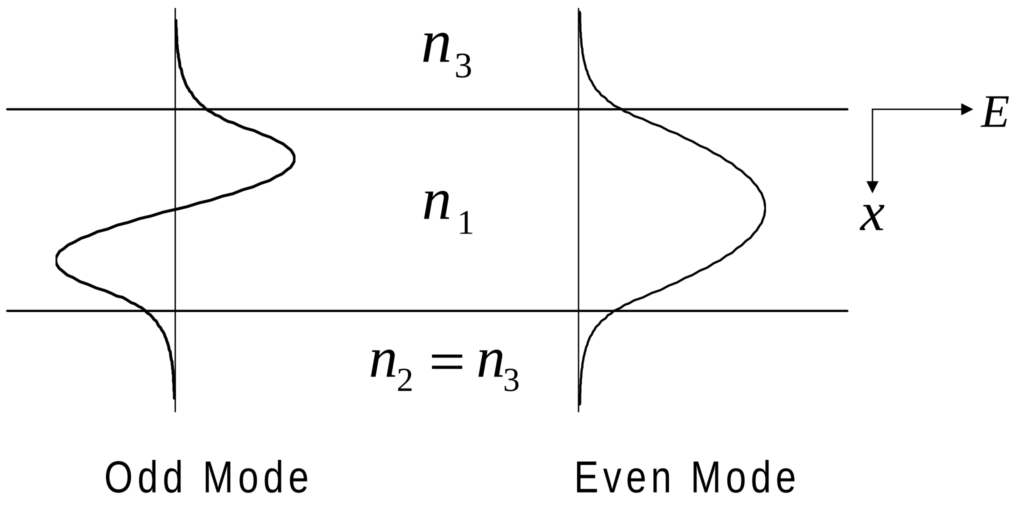

Figure 2.3: Electric Field Distribution in a Symmetric Slab Waveguide

2.4 The Wave Equation for a Slab Waveguide

Consider the asymmetric slab waveguide shown in figure 2.2. Maxwell's equations can be written in terms of the refractive index n_i $(i = 1, 2, 3)$ of the three layers and by assuming that the material of each layer is non-magnetic and isotropic, that is $\mu = \mu_0$ and ϵ is a scalar, we have [39, 40, 41]:

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \tag{2.5}$$

$$\nabla \times \mathbf{H} = n_i^2 \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{2.6}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{2.7}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{2.8}$$

To obtain the above equations, we used $\mathbf{D} = \epsilon \mathbf{E} = n^2 \epsilon_o \mathbf{E}$, $\mathbf{B} = \mu_o \mathbf{H}$, $\mathbf{J} = 0$, and

$\rho = 0$ in equations 2.1 to 2.4.

If we apply the curl operator to equation 2.5, we get:

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \nabla \times \frac{\partial \mathbf{H}}{\partial t} \tag{2.9}$$

$$= -\mu_o n_i^2 \epsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.10}$$

where equation 2.6 has been used to eliminate $\mathbf{H}$. To simplify further, we use the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{2.11}$$

where $\mathbf{A}$ is an arbitrary vector field. Using equations 2.7 and 2.11, equation 2.10 can be simplified to:

$$\nabla^2 \mathbf{E} = \mu_o \epsilon_o n_i^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.12}$$

Writing the above equation in phasor notation (assuming a time-harmonic field of the form $e^{-j\omega t}$) we obtain [39, 40]:

$$\nabla^2 \mathbf{E} + k_o^2 n_i^2 \mathbf{E} = 0 \tag{2.13}$$

which is the familiar three-dimensional vector wave equation for a uniform dielectric with refractive index n_i. Here k_o is the free-space wave number given by $k_o = \omega\sqrt{\mu_o \epsilon_o}$. The electric field vector $\mathbf{E}$ in equation 2.13 is a phasor quantity, which is complex and has both a magnitude and a phase. In addition, $\mathbf{E}$ is in general a function of space co-ordinates x, y, z and angular frequency ω. $\mathbf{E}$ is independent of time since the time dependence has been removed by the phasor transformation.

We may simplify equation 2.13 by assuming that the structure is uniform in the y-direction (see figure 2.1) and extends to infinity in the y-direction. This allows us to assume that the field **E** is also uniform in this direction. Thus $\frac{\partial}{\partial y}$ is replaced by zero. If we further assume a z-dependence of the form $e^{j\beta z}$, with β as the longitudinal propagation constant, equation 2.13 is simplified and takes the form:

$$\frac{d^2\mathbf{E}}{dx^2} + (k_o^2 n_i^2 - \beta^2)\mathbf{E} = 0 \tag{2.14}$$

The above equation is known as Helmholtz equation. In this case **E** is a function of x only and the equation is a second order ordinary differential equation. The propagation constant β can be expressed as $\beta = k_o n_{eff}$, where n_{eff} is called the effective index. The field of a slab waveguide is in general a superposition of Transverse Electric (TE) polarized field and Transverse Magnetic (TM) polarized field. The field components of the two polarizations are H_x, E_y and H_z for TE-polarized waves and E_x, H_y and E_z for TM-polarized waves.

2.5 Transverse Electric (TE) Guided Modes

By using equation 2.14, the TE scalar wave equation for the three waveguide regions takes the following form:

$$\frac{d^2E_y}{dx^2} - r^2E_y = 0 \quad , \quad x \le 0 \tag{2.15}$$

$$\frac{d^2E_y}{dx^2} + q^2E_y = 0 \quad , \quad 0 \le x \le 2d \tag{2.16}$$

$$\frac{d^2E_y}{dx^2} - p^2E_y = 0 \quad , \quad x \ge 2d \tag{2.17}$$

where $r^2 = \beta^2 - k_o^2 n_3^2$, $q^2 = k_o^2 n_1^2 - \beta^2$ and $p^2 = \beta^2 - k_o^2 n_2^2$. For guided modes, we require that the power to be confined largely to the central region of the guide and no power escapes from the structure. The form of equations 2.15, 2.16 and 2.17 then implies that this requirement will be satisfied for an oscillatory solution in the core region ($q^2 \geq 0$) with evanescent tails in the cladding and substrate regions ($r^2, p^2 \geq 0$) (see figure 2.4). Assuming $n_1 > n_2 \geq n_3$, it is straightforward to show that for guided modes, the possible range of β is given by $k_o n_1 \geq \beta \geq k_o n_2 \geq k_o n_3$.

From equation 2.5, the other field components of the TE modes are obtained in terms of E_y as follows:

$$H_x = -\frac{\beta}{\omega\mu_o} E_y \tag{2.18}$$

$$H_z = -\frac{j}{\omega\mu_o}\frac{\partial E_y}{\partial x} \tag{2.19}$$

Thus, for guided modes the solution of E_y in the three regions is [39, 40]:

$$E_y = \begin{cases} Ae^{rx} & , x \leq 0 \\ A\cos(qx) + B\sin(qx) & , 0 \leq x \leq 2d \\ \left(A\cos(2dq) + B\sin(2dq)\right) e^{-p(x-2d)} & , x \geq 2d \end{cases} \tag{2.20}$$

where A and B are constants. By examining equation 2.20, the boundary condition on E_y is satisfied by its continuity at both $x = 0$ and $x = 2d$. The other tangential field component to the waveguide interfaces, namely H_z, must also be continuous at

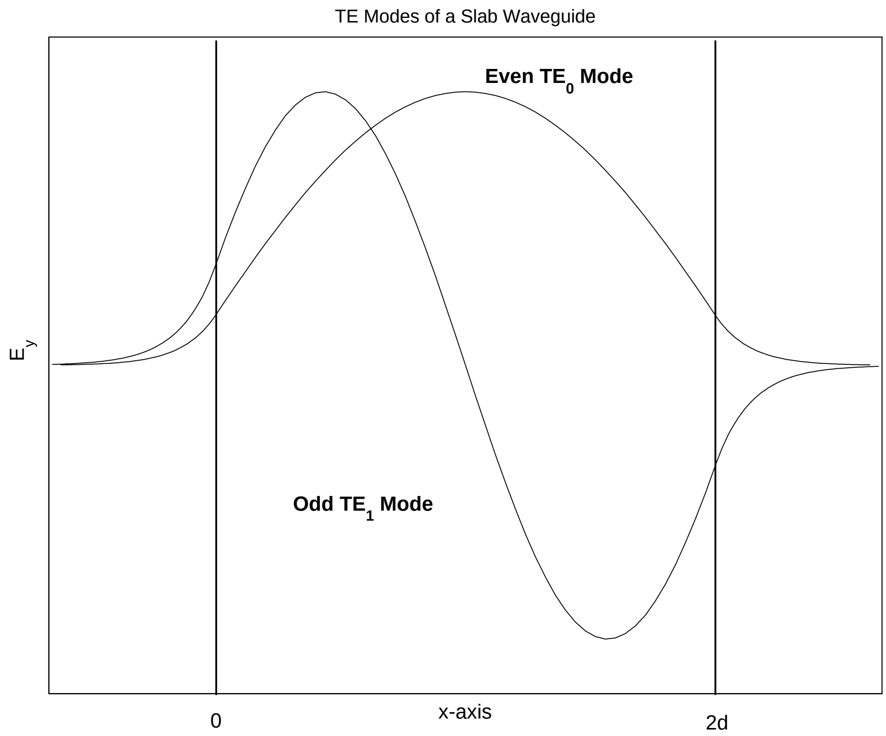

Figure 2.4: TE Mode Patterns of a Slab Waveguide

these interfaces. From equations 2.19 and 2.20, we have:

$$H_z = \frac{-j}{\omega\mu_o} \begin{cases} rAe^{rx} & ,x \leq 0 \\ q\left(-A\sin(qx) + B\cos(qx)\right) & ,0 \leq x \leq 2d \\ -p\left(A\cos(2dq) + B\sin(2dq)\right)e^{-p(x-2d)} & ,x \geq 2d \end{cases} \tag{2.21}$$

The continuity condition of H_z yields two equations. One at $x = 0$ and the second at $x = 2d$, that is:

$$rA = qB \tag{2.22}$$

and

$$q\left(-A\sin(2dq) + B\cos(2dq)\right) = -p\left(A\cos(2dq) + B\sin(2dq)\right) \tag{2.23}$$

Eliminating the ratio A/B from these equations yields [39, 40]:

$$\tan(2dq) = \frac{q(p+r)}{q^2 - pr} \tag{2.24}$$

This is the eigenvalue equation for the TE modes of the asymmetric slab waveguide. Equation 2.24 is an implicit relationship which involves the wavelength, refractive indices of the layers and core thickness as known quantities, and the propagation constant β as the only unknown quantity. It can be shown that only certain discrete values of β can satisfy the above equation, so this waveguide will only support a *discrete* set of guided modes. The allowed values of β can be found from this equation using numerical or graphical methods. After evaluating β, the previous equations are used to obtain the modal field in each layer. The symmetric waveguide ($n_2 = n_3$) can only support modes with even or odd electric field patterns. In this case it can be easily shown that the eigen-value equation 2.24 reduces to ($p = r$):

$$\tan(2dq) = \frac{2pq}{q^2 - p^2} \tag{2.25}$$

An example of the field pattern of the TE modes for a three-layer slab waveguide is given in figure 2.4.

2.6 Transverse Magnetic (TM) Guided Modes

The wave equation for this polarization is obtained in terms of the magnetic field component H_y as:

$$\frac{d^2 H_y}{dx^2} - r^2 H_y \quad = \quad 0 \quad , \quad x \leq 0 \tag{2.26}$$

$$\frac{d^2 H_y}{dx^2} + q^2 H_y = 0 \quad , \quad 0 \le x \le 2d \tag{2.27}$$

$$\frac{d^2 H_y}{dx^2} - p^2 H_y = 0 \quad , \quad x \ge 2d \tag{2.28}$$

From equation 2.6, the other field components of the TM modes are obtained in terms of H_y as:

$$E_x = \frac{\beta}{\omega n_i^2 \epsilon_o} H_y \tag{2.29}$$

$$E_z = \frac{j}{\omega n_i^2 \epsilon_o} \frac{\partial H_y}{\partial x} \tag{2.30}$$

Thus, the solution of H_y in the three regions for the guided modes is [39, 40]:

$$H_y = \begin{cases} C e^{rx} & , x \le 0 \\ C\cos(qx) + D\sin(qx) & , 0 \le x \le 2d \\ \left(C\cos(2dq) + D\sin(2dq)\right) e^{-p(x-2d)} & , x \ge 2d \end{cases} \tag{2.31}$$

where C and D are constants. The field component E_z is obtained from equations 2.30 and 2.31 as follows:

$$E_z = \frac{j}{\omega \epsilon_o} \begin{cases} \frac{rC}{n_3^2} e^{rx} & , x \le 0 \\ \frac{q}{n_1^2}\left(-C\sin(qx) + D\cos(qx)\right) & , 0 \le x \le 2d \\ \frac{-p}{n_2^2}\left(C\cos(2dq) + D\sin(2dq)\right) e^{-p(x-2d)} & , x \ge 2d \end{cases} \tag{2.32}$$

Continuity of E_z at $x = 0$ and $x = 2d$ leads to:

$$\frac{rC}{n_3^2} = \frac{qD}{n_1^2} \tag{2.33}$$

and

$$\frac{q}{n_1^2}\left(-C\sin(2dq) + D\cos(2dq)\right) = \frac{-p}{n_2^2}\left(C\cos(2dq) + D\sin(2dq)\right) \tag{2.34}$$

Eliminating the ratio C/D from these two equations results in [39, 40]:

$$\tan(2dq) = \frac{qn_1^2\left(n_3^2 p + n_2^2 r\right)}{n_2^2 n_3^2 q^2 - n_1^4 pr} \tag{2.35}$$

which is the eigenvalue equation for TM modes of an asymmetric slab waveguide. An example of the TM mode patterns for a symmetric slab waveguide is given in figure 2.5. As evident from the figure, H_y is continuous across a layer interface but its derivative is discontinuous there, causing a sudden change in the slope of H_y there.

2.7 Mode Numbers and Cut-Offs

The notation TE_N (and similarly TM_N) is used to refer to a mode possessing N nodes in the distribution of E_y for TE modes and H_y for TM modes. The value of N can be obtained by taking the argument of the tangent in the eigenvalue equations 2.24 and 2.35 to be $(2dq - N\pi)$. Since $n_1 > n_2 > n_3$, the cut-off condition is given by [39]:

$$\beta = k_o n_2 \tag{2.36}$$

This corresponds to loss of optical confinement due to loss of exponential decay away from the waveguide in the substrate. The resultant effect is a field-spreading throughout the substrate region.

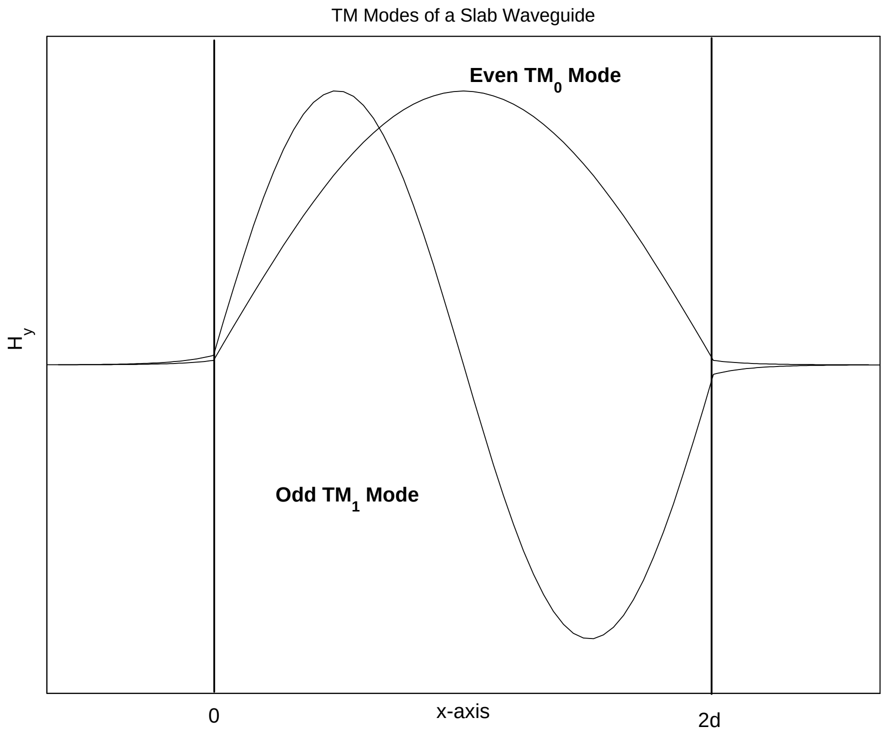

Figure 2.5: TM Mode Patterns of a Slab Waveguide

The cut-off conditions for TE_N and TM_N modes can be found by using the above definitions for the mode numbers and cut-offs. Substituting equation 2.36 into equation 2.24 along with the appropriate expressions for p, q, r at cut-off, the cut-off condition for the TE modes is stated as [39]:

$$\tan(2dk_c(n_1^2 - n_2^2)^{1/2} - N\pi) = \left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2} \tag{2.37}$$

where k_c corresponds to the cut-off wave number for TE_N. In terms of the normalized frequency (v), given by:

$$v = k_o d(n_1^2 - n_2^2)^{1/2} \tag{2.38}$$

the cut-off value v_c for the TE_N mode is [39]:

$$v_c = \frac{1}{2}\tan^{-1}\left[\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right] + \frac{N\pi}{2} \tag{2.39}$$

where $\tan^{-1}$ is restricted to the range $0 - \pi/2$. Equation 2.39 can be used to obtain M, the number of TE guided modes and is found to be [39]:

$$M = \left\{\frac{1}{\pi}\left(2v - \tan^{-1}\left[\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right]\right)\right\}_{int} \tag{2.40}$$

where the subscript int indicates the next largest integer.

The corresponding cut-off condition and number of guided TM modes are given as follows [39]:

$$v_c = \frac{1}{2}\tan^{-1}\left[\left(\frac{n_1}{n_3}\right)^2\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right] + \frac{N\pi}{2} \tag{2.41}$$

$$M = \left\{\frac{1}{\pi}\left(2v - \tan^{-1}\left[\left(\frac{n_1}{n_3}\right)^2\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right]\right)\right\}_{int} \tag{2.42}$$