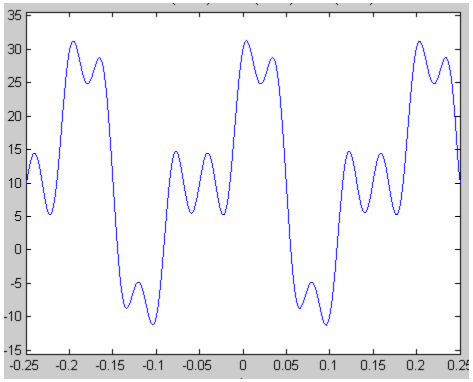
For the periodical signal

$f_1 = 5$ HZ $f_2 = 10$ HZ $f_3 = 25$ HZ

$$f(t) = 10 + 14\cos(10\pi t) + 10\sin(20\pi t) + 6\cos(50\pi t)$$



$$f(t) = 10 + 14\left[\frac{e^{j10\pi t} + e^{-j10\pi t}}{2}\right] + 10\left[\frac{e^{j20\pi t} - e^{-j20\pi t}}{2j}\right] + 6\left[\frac{e^{j50\pi t} + e^{-j50\pi t}}{2}\right]$$

$$f(t) = 10 + \left[7e^{j10\pi t} + 7e^{-j10\pi t}\right] + \left[\frac{5}{j}e^{j20\pi t} - \frac{5}{j}e^{-j20\pi t}\right] + \left[3e^{j50\pi t} + 3e^{-j50\pi t}\right]$$

$$f(t) = 3e^{-j50\pi t} + 5je^{-j20\pi t} + 7e^{-j10\pi t} + 10 + 7e^{j10\pi t} - 5je^{j20\pi t} + 3e^{j50\pi t}$$

$f_1 = 5\ HZ$ $f_2 = 10\ HZ$ $f_3 = 25\ HZ$ → $f_0 = 5\ HZ$

$\omega_0 = 10\pi$

$$f(t) = 10 + 14\cos(10\pi t) + 10\sin(20\pi t) + 6\cos(50\pi t)$$

$$f(t) = 3e^{-j50\pi t} + 5je^{-j20\pi t} + 7e^{-j10\pi t} + 10 + 7e^{j10\pi t} - 5je^{j20\pi t} + 3e^{j50\pi t}$$

C_{-5} C_{-2} C_{-1} C_0 C_1 C_2 C_5

The subscript on the coefficients is determined as follows :

The angular frequency associated with is 10π **(the fundamental)**

$7e^{-j10\pi t}$ → 7 is C_1 and $C_{-1} = 7$

The angular frequency associated with is 20π ***which is twice the fundamental***

$-5je^{-j20\pi t}$ → $-5j$ is C_2 and $C_{-2} = 5j$

The angular frequency associated with is 50π ***which is five time the fundamental***

$3e^{-j50\pi t}$ → 3 is C_5 and $C_{-5} = 3$

$f_1 = 5\text{ HZ}$ $\quad f_2 = 10\text{ HZ}$ $\quad f_3 = 25\text{ HZ}$ $\quad \rightarrow \quad f_0 = 5\text{ HZ}$

$$f(t) = 10 + 14\cos(10\pi t) + 10\sin(20\pi t) + 6\cos(50\pi t)$$

$\omega_0 = 10\pi$

$$f(t) = 3e^{-j50\pi t} + 5je^{-j20\pi t} + 7e^{-j10\pi t} + 10 + 7e^{j10\pi t} - 5je^{j20\pi t} + 3e^{j50\pi t}$$

C_{-5} $\quad C_{-2}$ $\quad C_{-1}$ $\quad C_0$ $\quad C_1$ $\quad C_2$ $\quad C_5$

$$\begin{aligned} f(t) = 10 & \\ & + 14\cos(10\pi t) \\ & + 10\sin(20\pi t) \\ & + 6\cos(50\pi t) \end{aligned}$$

Time Domain

$\Longleftrightarrow$

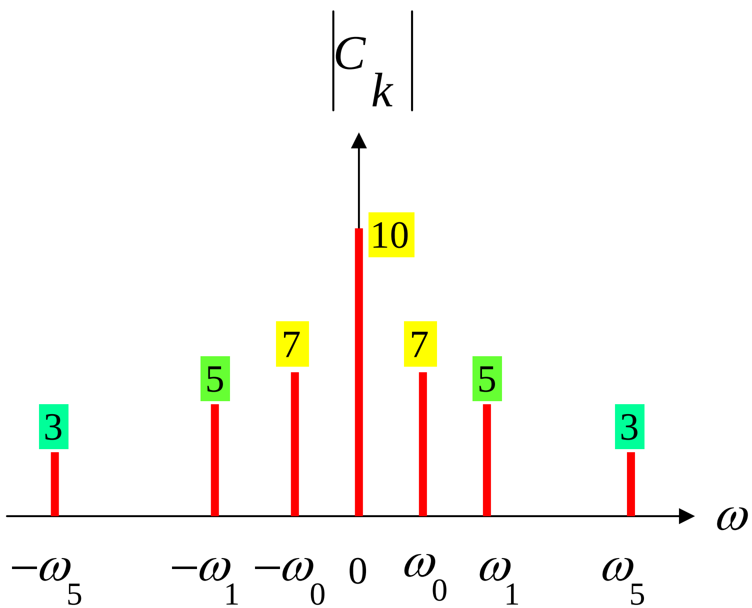
C_0, C_1, C_2, C_5

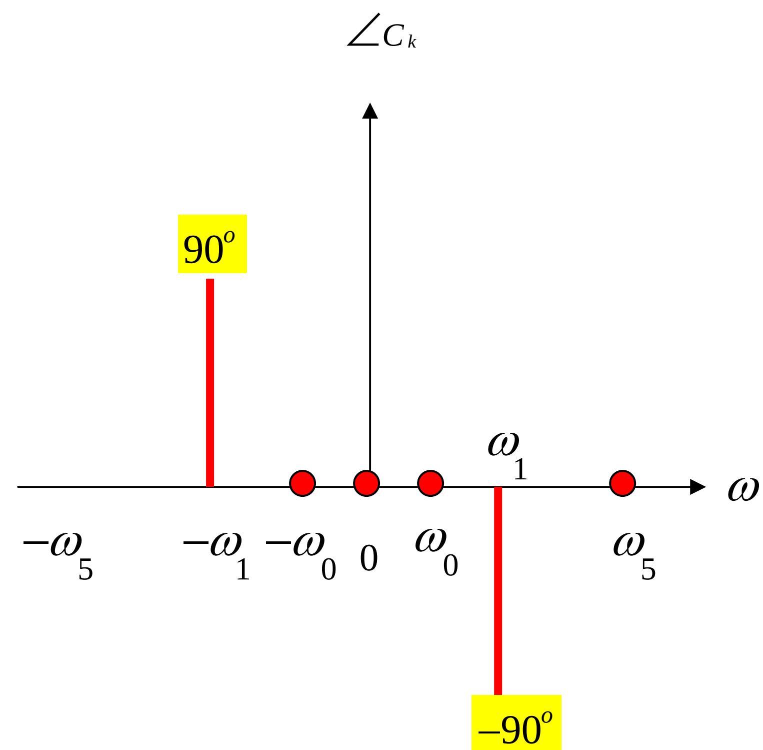
$\omega_0 = 10\pi$

Frequency Domain

$$f(t) = 3e^{-j50\pi t} + 5je^{-j20\pi t} + 7e^{-j10\pi t} + 10 + 7e^{j10\pi t} - 5je^{j20\pi t} + 3e^{j50\pi t}$$

C_{-5} C_{-2} C_{-1} C_0 C_1 C_2 C_5

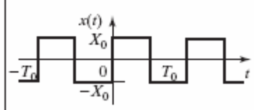
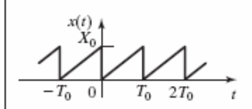
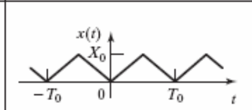
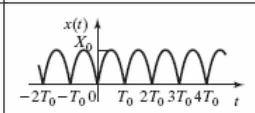
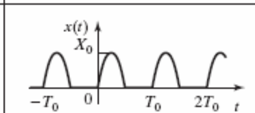
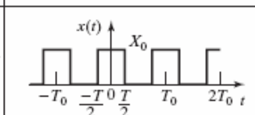
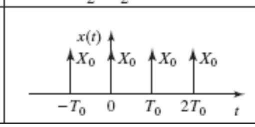




FOURIER SERIES TRANSFORMATIONS

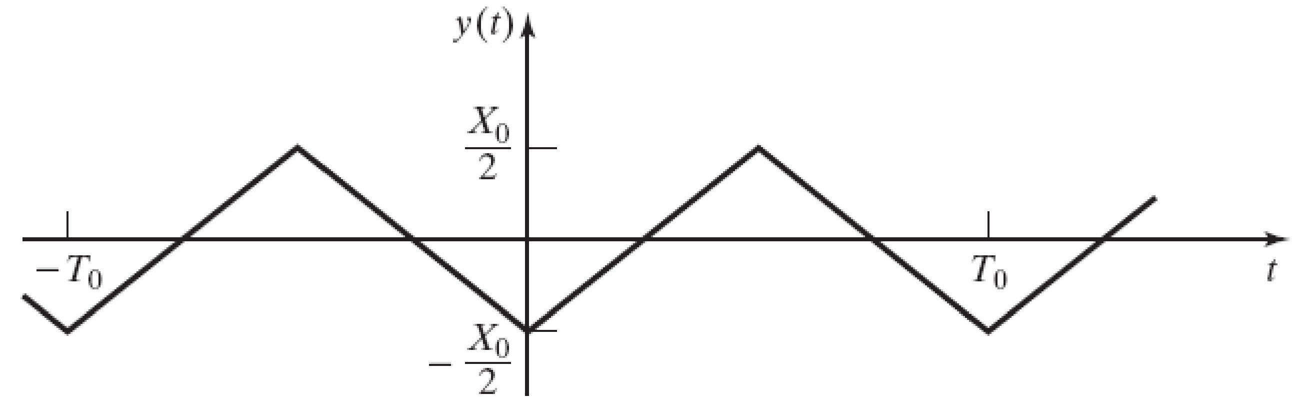
Table 4.3 gives the Fourier coefficients for seven common signals.

TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\,\mathrm{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

We now give two procedures that extend the usefulness of this table.

Suppose we want to find the Fourier Series complex coefficients for the periodical signal y(t)



Method 1 we can use $$C_{yk} = \frac{1}{T_0} \int_{T_0} y(t) e^{-jk\omega_0 t} dt$$

Question : can we find the coefficients of y(t) without using the integration formula ?

Answer : Let us try the table 4-3 ?

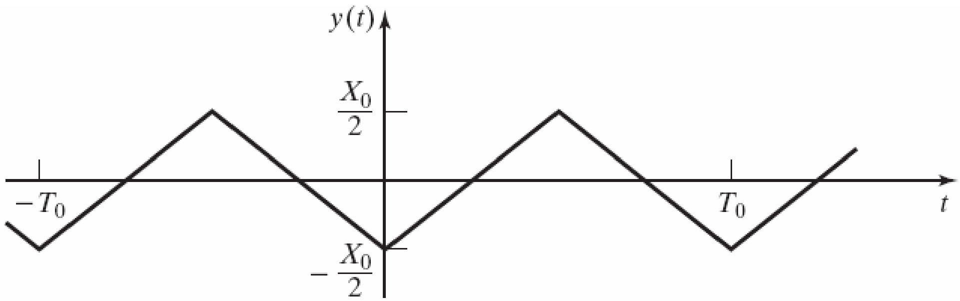
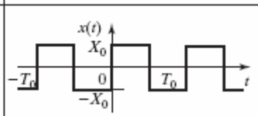
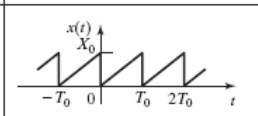
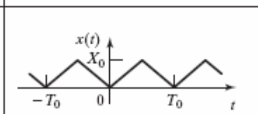
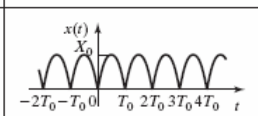
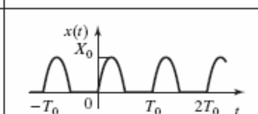
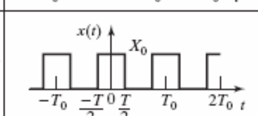
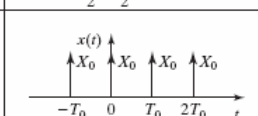


TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\text{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

Unfortunately there is no function
In the table that is identical to y(t)

And it shouldn't ? Why ?

That will require a table of an infinite length to satisfies all possible periodical function

An impossible task !

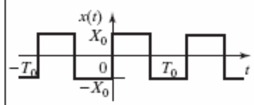
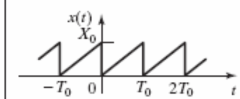
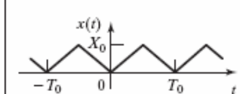
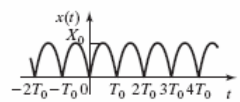
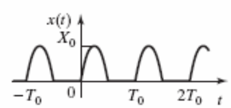
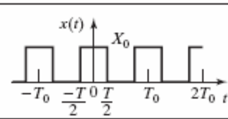
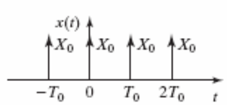
So how can we use the table (known Coefficients)
To find the Coefficient of periodical function ?

Next section will explain that

FOURIER SERIES TRANSFORMATIONS

Table 4.3 gives the Fourier coefficients for seven common signals.

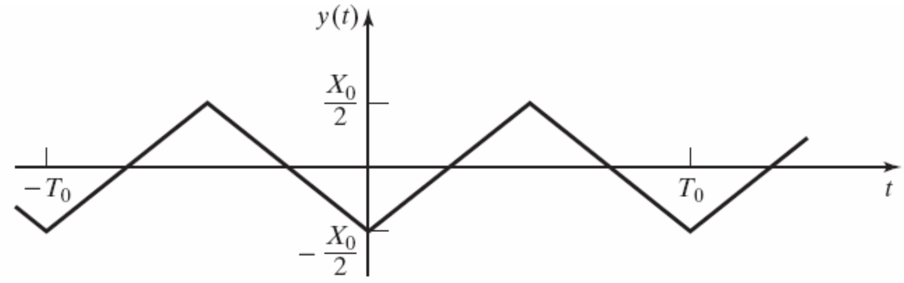
TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\,\text{sinc}\,\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

We now give two procedures that extend the usefulness of this table.

Amplitude Transformations

If we are given the signal y(t)



$$y(t) = C_{0y} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_{ky} e^{jk\omega_0 t}$$

If we can find $x(t) = C_{0x} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t}$ such that $y(t) = Ax(t) + B$ were A and B are constants

Then

$$y(t) = C_{0y} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_{ky} e^{jk\omega_0 t} = A\left[C_{0x} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t}\right] + B$$

$$= \underbrace{[AC_{0x} + B]}_{C_{0y}} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \underbrace{AC_{kx}}_{C_{ky}} e^{jk\omega_0 t}$$

$$C_{0y} = AC_{0x} + B$$

$$C_{ky} = AC_{kx} \quad k \neq 0$$

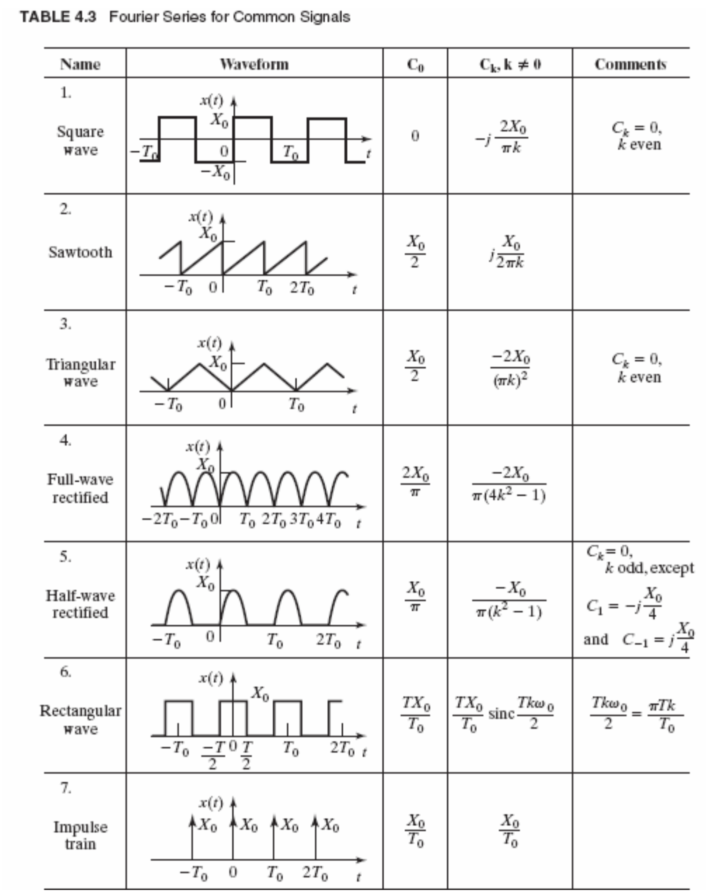
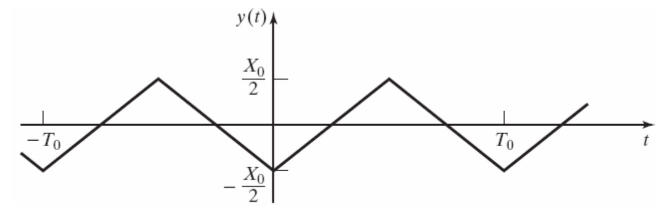


TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\operatorname{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	



This function y(t) resemble

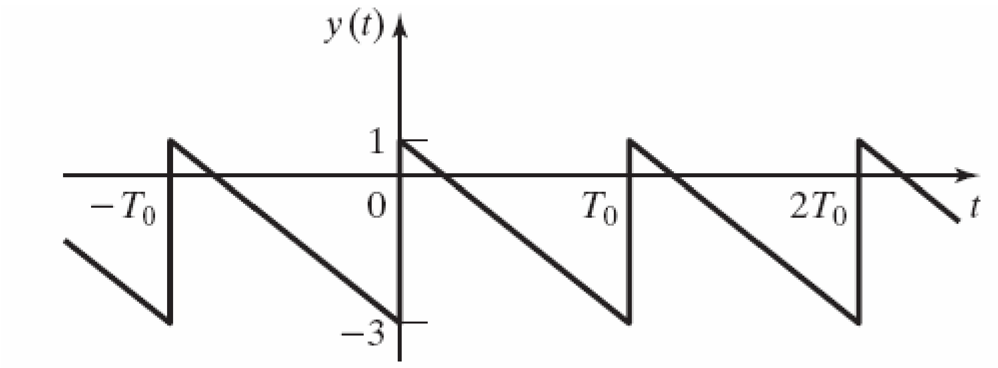
$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\ \text{odd}}}^{\infty} \frac{-2X_0}{(\pi k)^2} e^{jk\omega_0 t}$$

$$y(t) = x(t) - \frac{X_0}{2} = \sum_{\substack{k=-\infty \\ k\ \text{odd}}}^{\infty} \frac{-2X_0}{(\pi k)^2} e^{jk\omega_0 t}$$

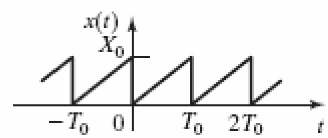
$$= C_{0y} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_{ky} e^{jk\omega_0 t}$$

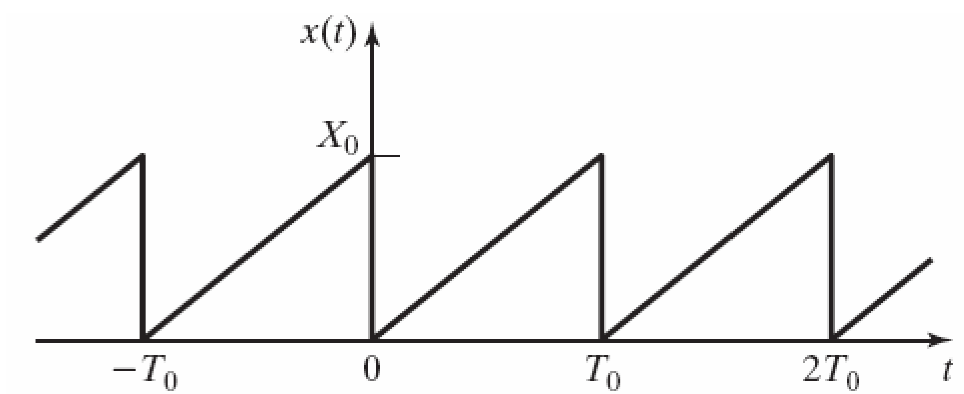
$$C_{0y} = 0 \quad C_{ky} = \frac{-2X_0}{(\pi k)^2}$$

Example 4.9 Find Fourier Series complex coefficients for the periodical signal y(t)

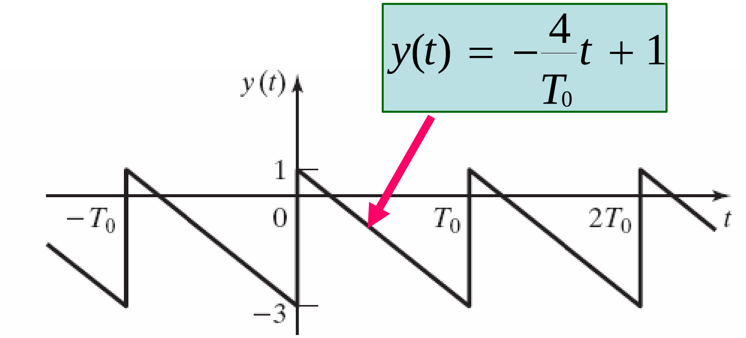


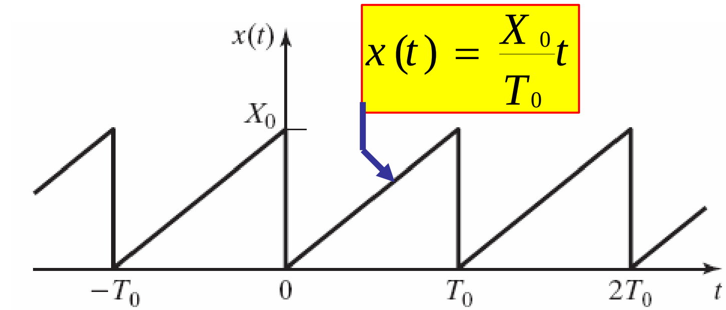
From Table 4-3 we have

2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	



$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{X_0}{2\pi k} e^{j\pi/2} e^{jk\omega_0 t}$$





$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{X_0}{2\pi k} e^{j\pi/2} e^{jk\omega_0 t}.$$

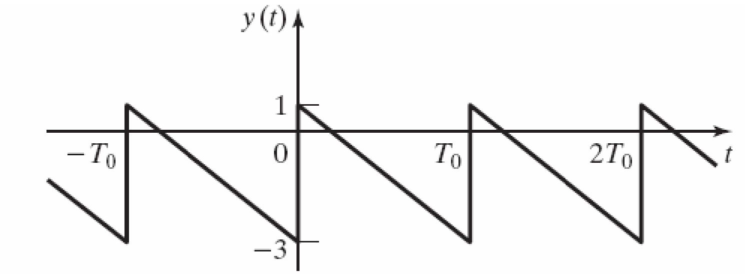
$$y(t) = Ax(t) + B$$

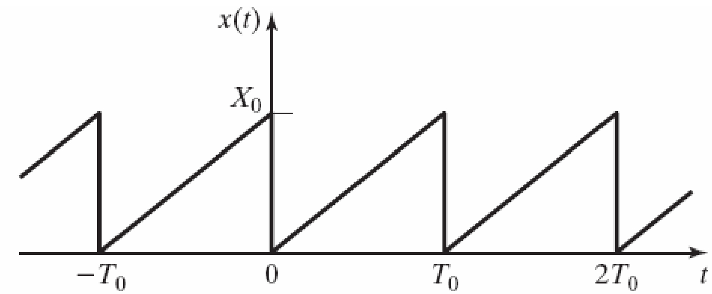
So what is A and B ?

Since $x(t) = \frac{X_0}{T_0}t \quad \Rightarrow \quad y(t) = A\frac{X_0}{T_0}t + B$

Since $y(t) = -\frac{4}{T_0}t + 1 \quad \Rightarrow \quad A\frac{X_0}{T_0}t + B = -\frac{4}{T_0}t + 1$

$$\Rightarrow \quad A\frac{X_0}{T_0} = -\frac{4}{T_0} \quad \Rightarrow A = -\frac{4}{X_0} \qquad B = 1$$





$$y(t) = Ax(t) + B \qquad A = -\frac{4}{X_0} \qquad B = 1 \qquad x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{X_0}{2\pi k} e^{j\pi/2} e^{jk\omega_0 t}$$

$$y(t) = C_{0y} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{ky} e^{jk\omega_0 t} = A\left[C_{0x} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t}\right] + \mathrm{B}$$

$$= \underbrace{\left[AC_{0x} + \mathrm{B}\right]}_{C_{0y}} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \underbrace{AC_{kx}}_{C_{ky}} e^{jk\omega_0 t}$$

$$C_{0y} = AC_{0x} + \mathrm{B}$$

$$C_{ky} = AC_{kx} \qquad k \neq 0$$

$$C_{0y} = \left(-\frac{4}{X_0}\right)\left(\frac{X_0}{2}\right) + 1 = -1$$

$$\underset{k\neq 0}{C_{ky}} = AC_{kx} = \left(-\frac{4}{X_0}\right)\left(\frac{X_0}{2\pi k} e^{j\pi/2}\right)$$

$$\underset{k\neq 0}{C_{ky}} = \frac{2}{\pi k} e^{j\pi/2}$$