Problem 1 [~~40~~ pts]

(a) $$X_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{5}{2}\int_0^2 t\, e^{-jn\omega_0 t}\, dt \quad ; \text{ Given } \int t e^{\alpha t} dt = \frac{e^{\alpha t}}{\alpha^2}(\alpha t - 1) + C$$

$$= \frac{5}{2}\left[\frac{e^{-jn\pi t}}{-(n\pi)^2}\left\{-jn\pi t - 1\right\}\Big|_0^2 \right] \quad ; \omega_0 = \pi$$

$$= \frac{5}{2}\left[\frac{e^{-j2n\pi}}{(n\pi)^2}\left\{j2n\pi + 1\right\} - \frac{1}{(n\pi)^2} \right] \quad ; \ e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1.$$

$$= \frac{5}{\cancel{2}}\left[\frac{j\cancel{2}\cancel{n}\cancel{\pi}}{n^2\pi^2} \right] = \frac{j5}{n\pi}$$

(b) $$X_0 = \frac{\text{Area of one Period}}{\text{Period}} = \frac{\frac{1}{2}\times 2\times 10}{2} = 5.$$

(c) $$\text{Power of the Signal} = \frac{1}{2}\int_0^2 (5t)^2 dt = \frac{25}{2}\times \frac{t^3}{3}\Big|_0^2$$

$$= \frac{25}{6}\times 8 = \frac{100}{3}.$$

$$\therefore \sum_{n=-\infty}^{\infty} |X_n|^2 = \frac{100}{3} = |X_0|^2 + 2\sum_{n=1}^{\infty}|X_n|^2$$

$$\therefore \sum_{n=1}^{\infty}|X_n|^2 = \left(\frac{100}{3} - 25\right)\times\frac{1}{2} = \frac{25}{6}.$$

(d) $$a_3 = 2\,\mathrm{Re}(X_3) = 0.$$

(e) $$X_0 = 5 \Rightarrow Y_0 = X_0 H(0) = 5\times 1 = 5$$

$$X_1 = j\frac{5}{\pi} \Longrightarrow Y_1 = \left(j\frac{5}{\pi}\right) \times \left(\frac{1}{1+j\pi}\right) = \frac{j5/\pi}{1+j\pi}$$

Problem 2 [25 points]

The signal $x(t)$ has the Fourier Transform $X(f) = \dfrac{1}{1+j2\pi f} + \dfrac{1}{(3+j2\pi f)^2}$. Find the Fourier Transform of each of the following:

(a) $x(\frac{t-2}{4})$

(b) x(t)cos(20πt)

(c) x(t) * δ(t-3) (* is the Convolution Process)

(a) $\mathcal{F}\left[x\left(\frac{t}{4}\right)\right] = \dfrac{4}{1+j8\pi f} + \dfrac{4}{(3+j8\pi f)^2}$

$$\mathcal{F}\left[x\left(\frac{t-2}{4}\right)\right] = \left[\frac{4}{1+j8\pi f} + \frac{4}{(3+j8\pi f)^2}\right] e^{-j4\pi f}$$

(b) $\mathcal{F}[x(t)\cos 20\pi t]$

$$= \frac{1/2}{1+j2\pi(f-10)} + \frac{1/2}{[3+j2\pi(f-10)]^2}$$

$$+ \frac{1/2}{1+j2\pi(f+10)} + \frac{1/2}{[3+j2\pi(f+10)]^2}$$

(c) $x(t) * \delta(t-3) \Leftrightarrow X(f)\cdot e^{-j6\pi f}$

$$= \left(\frac{1}{1+j2\pi f} + \frac{1}{(3+j2\pi f)^2}\right) e^{-j6\pi f}$$

Problem 3 [35 pts]

Consider the signal $x(t) = \Pi(5t)$.

(a) Sketch the signal x(t) and its derivative $dx(t)/dt$.

(b) Using ONLY the Fourier Transform $\delta(t) \leftrightarrow 1$ and the Differentiation Property $dx(t)/dt \leftrightarrow (j2\pi f)\, X(f)$, find the FT of the signal $x(t) = \Pi(5t)$.

(c) Sketch the amplitude spectrum of $x(t)$. Show all significant points.

(d) Find the Energy Spectral Density of $x(t)$.

(e) If the signal $x(t)$ is fed to an ideal LPF of cutoff frequency 10 Hz, find the energy at the output of the filter (Write the expression. Do not evaluate it).

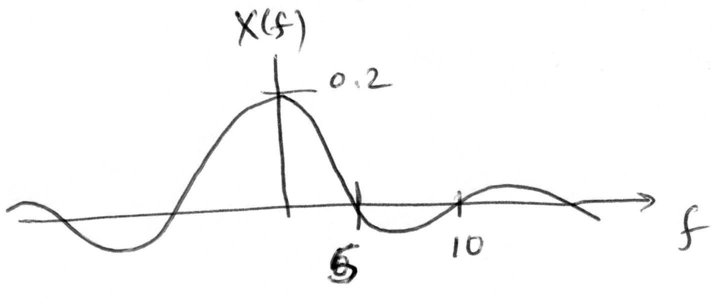
(a)

(b)

$$\frac{dx(t)}{dt} = \delta(t+0.1) - \delta(t-0.1) \Leftrightarrow e^{j0.2\pi f} - e^{-j0.2\pi f} = (j2\pi f)\,X(f)$$

$$\therefore X(f) = \frac{e^{j0.2\pi f} - e^{-j0.2\pi f}}{j2\pi f} = 0.2\,\mathrm{sinc}\left(\frac{f}{5}\right)$$

(c)



(d) $0.04\,\mathrm{sinc}^2\left(\frac{f}{5}\right)$

(e) $\displaystyle\int_{-10}^{10} 0.04\,\mathrm{sinc}^2\left(\frac{f}{5}\right) df.$