Verification of Carson's Rule (for computing the FM Bandwidth)

The Casron's rule says that the BW of any FM signal is given by

$$BW_{FM} \approx 2 \cdot \Delta f + 2B_m \quad \text{(Hz)} \approx 2\left[\Delta f + B_m\right] \quad \text{(Hz)} ,$$
$$\approx 2\left[\Delta\omega + 2\pi B_m\right] \quad \text{(rad/s)}$$

where

B_m = Bandwidth of the Message Signal $m(t)$ in Hz,

and

$$\Delta f = \Delta\omega/2\pi = k_f \cdot m_p/2\pi .$$

We can verify the Carson's rule using a simple message signal that is a sinusoid with frequency ω_m. That is

$$m(t) = \alpha \cdot \cos(\omega_m t) ,$$

where the magnitude of the message signal α is a constant. The bandwidth of this signal B_m is clearly ω_m. The signal $a(t)$, which is the integration of $m(t)$, becomes

$$a(t) = \int_{-\infty}^{t} m(\alpha)d\alpha = \frac{\alpha}{\omega_m} \cdot \sin(\omega_m t) .$$

So, the FM signal becomes

$$g_{FM}(t) = A \cdot \cos\left[\omega_c t + \frac{k_f \alpha}{\omega_m}\sin(\omega_m t)\right] .$$

Let us define a function $\hat{g}_{FM}(t)$ that is the complex–exponential form of $g_{FM}(t)$, or

$$\hat{g}_{FM}(t) = A \cdot e^{j\left[\omega_c t + \frac{k_f \alpha}{\omega_m}\sin(\omega_m t)\right]} = A \cdot e^{j\omega_c t} \cdot e^{j\frac{k_f \alpha}{\omega_m}\sin(\omega_m t)} . \qquad (1)$$

The second exponential above is a periodic signal with period $T_m = 2\pi/\omega_m$. So, it can be expanded using the exponential Fourier series as

$$e^{j\frac{k_f \alpha}{\omega_m}\sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_m t} . \qquad (2)$$

where according the exponential Fourier series expansion (page 53 in your text book)

$$D_n = \frac{1}{T_m}\int_{T_m} f(t)\cdot e^{-jn\omega_m t}dt = \frac{\omega_m}{2\pi}\int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} e^{j\frac{k_f\alpha}{\omega_m}\sin(\omega_m t)}e^{-jn\omega_m t}dt .$$

This integration can be simplified if we use the substitution $x = \omega_m t$

$$x = \omega_m t \quad \rightarrow \quad dx = \omega_m \cdot dt \quad \rightarrow \quad dt = \frac{1}{\omega_m}\cdot dx$$

t coveres the range $\left[-\dfrac{\pi}{\omega_m},\dfrac{\pi}{\omega_m}\right]$ $\quad\rightarrow\quad$ x coveres the range $\left[-\pi,\pi\right]$

Let us define β to be $\beta = \dfrac{\Delta\omega}{\omega_m} = \dfrac{k_f\alpha}{\omega_m}$. Substituting for β and for $x = \omega_m t$ in the integration of D_n, we get

$$D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\beta\sin(x)}e^{-jnx}dx .$$

Unfortunately, the above integration does not have a closed form (cannot be written in terms of basic functions). However, this function can be integrated numerically. The integration above is called the "Bessel function of the first kind with order n that is evaluated at β". This is abbreviated by Jn(β). So,

$$J_n(\beta) = D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\beta\sin(x)}e^{-jnx}dx .$$

One important property of this Bessel function is that

$$J_{-n}(\beta) = \left(-1\right)^n J_n(\beta) .$$

So, $\left|J_{-n}(\beta)\right| = \left|J_n(\beta)\right|$.

Substituting Jn(β) for Dn in Equation (2) above and using the result in Equation (1) gives

$$\hat{g}_{FM}(t) = A\cdot e^{j\omega_c t}\sum_{n=-\infty}^{\infty} J_n(\beta)\cdot e^{jn\omega_m t}$$

$$= A\sum_{n=-\infty}^{\infty} J_n(\beta)\cdot e^{j(\omega_c + n\omega_m)t} .$$

The original function $g_{FM}(t)$ is related to $\hat{g}_{FM}(t)$ by

$$g_{FM}(t) = \mathrm{Re}\left\{\hat{g}_{FM}(t)\right\} ,$$

So,

$$g_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cdot \cos\left[(\omega_c + n\omega_m)t\right],$$

where

$$\beta = \frac{\Delta\omega}{\omega_m} = \frac{k_f \alpha}{\omega_m}$$

is know as the MODULATION INDEX of the FM signal.
Conclusion:

The spectrum of an FM signal that results from modulating a sinusoidal message is an infinite sum of sinusoids with frequencies ωc, ωc±ωm, ωc±2ωm, ωc±3ωm, ωc±4ωm … . The amplitude of these sinusoids is the values of a Bessel function of the first kind that is evaluated at β.

Plotting the Bessel function of the first kind Jn(β) for different orders n and different values of β is shown below.
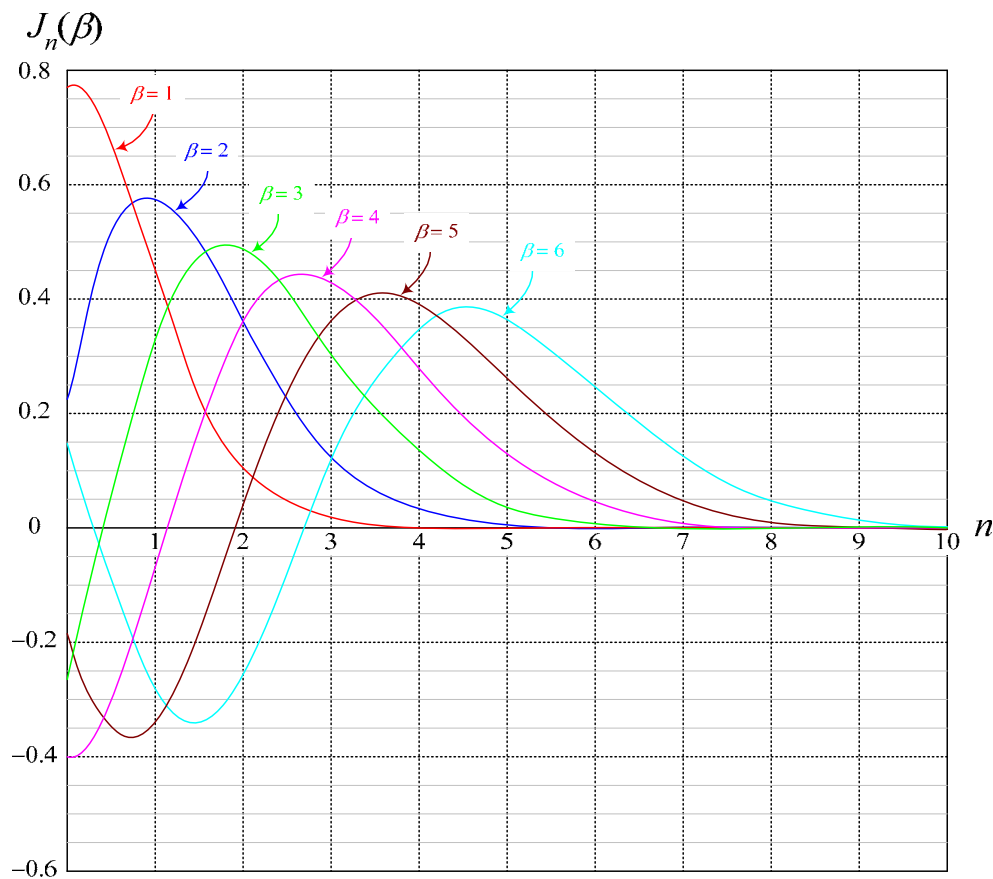
Looking at the above plots, we can fill in the following table

$J_n(\beta)$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
n = 0	0.7652	0.2239	–0.2601	–0.3971	–0.1776	0.1506
n = 1	0.4401	0.5767	0.3391	–0.0660	–0.3276	–0.2767
n = 2	0.1149	0.3528	0.4861	0.3641	0.0466	–0.2429
n = 3	0.0196	0.1289	0.3091	0.4302	0.3648	0.1148
n = 4	0.0025	0.0340	0.1320	0.2811	0.3912	0.3576
n = 5	0.0002	0.0070	0.0430	0.1321	0.2611	0.3621
n = 6	0.0000	0.0012	0.0114	0.0491	0.1310	0.2458
n = 7	0.0000	0.0002	0.0025	0.0152	0.0534	0.1296
n = 8	0.0000	0.0000	0.0005	0.0040	0.0184	0.0565
n = 9	0.0000	0.0000	0.0001	0.0009	0.0055	0.0212
n = 10	0.0000	0.0000	0.0000	0.0002	0.0015	0.0070

The power of the FM signal is A2 since it has a magnitude of A. Looking carefully at the table, we notice that the values of the elements below the thick line are all below (0.1). Therefore, the magnitude of each component that corresponds to one of these values is less than (0.1)A. So, the power of these components is [(0.1)A]2 = (0.01)A2 W. Since these components are on both sides around the carrier frequency, we see that the total power contained in all of these components is less than 0.02 W. This means that the power contained in the components above the thick line is at least 98% of the power of the FM signal. Therefore, we can safely ignore the components below that line.

It is seen from the table that for each value of β, only the components with $|n| \leq \beta + 1$ are significant and the rest are small. Eliminating all the components in Equation (3) with negligible amplitudes, the bandwidth of the FM signal becomes

$$BW_{FM} = 2\omega_m (\beta + 1) = 2(\omega_m \beta + \omega_m)$$
$$= 2(\Delta\omega + 2\pi B_m) \quad \text{(rad/s)}$$

This verifies that the BW of an FM signal is given by the Carson's rule given above.

Exercise: A sinusoidal message signal m(t) with a frequency of 5000 Hz and amplitude of 5 V is used to generate an FM signal with amplitude A = 10 V using a modulator with kf = 2π 3000 and a carrier frequency of 5 MHz. Find the following:
a)   The frequency deviation and the range of instantaneous frequency (find both in rad/s and in Hz) for the FM signal.
b)   The modulation index of the FM signal.
c)   The Bandwidth of the FM signal (both in rad/s and Hz).
d)   Sketch the magnitude spectrum of the FM signal and show that magnitudes of all the components within the bandwidth of the FM signal.

e)   Show that the power contained within the bandwidth of the FM signal is more than 98% of the total power of the FM signal.

f)   If we wanted the bandwidth of the FM signal to contain at least 99.8% of the FM signal power, how much would the bandwidth of the signal be?

e)   Repeat the above when the value of kf is doubled.