## Bandwidth of FM and PM Signals

The bandwidth of the different AM modulation techniques ranges from the bandwidth of the message signal (for SSB) to twice the bandwidth of the message signal (for DSBSC and Full AM). When FM signals were first proposed, it was thought that their bandwidth can be reduced to an arbitrarily small value. Compared to the bandwidth of different AM modulation techniques, this would in theory be a big advantage. It was assumed that a signal with an instantaneous frequency that changes over of range of $\Delta f$ Hz would have a bandwidth of $\Delta f$ Hz. When experiments were done, it was discovered that this was not the case. It was discovered that the bandwidth of FM signals for a specific message signal was at least equal to the bandwidth of the corresponding AM signal. In fact, FM signals can be classified into two types: Narrowband and Wideband FM signals depending on the bandwidth of each of these signals

## Narrowband FM and PM

The general form of an FM signal that results when modulating a signals $m(t)$ is

$$g_{FM}(t) = A \cdot \cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\alpha) d\alpha\right].$$

A narrow band FM signal satisfies the condition that

$$k_f \ll 1,$$

such that a change in the message signal does not results in a lot of change in the instantaneous frequency of the FM signal. For convenience, let

$$a(t) = \int_{-\infty}^{t} m(\alpha) d\alpha .$$

So,

$$g_{FM}(t) = A \cdot \cos\left[\omega_c t + k_f a(t)\right].$$

Important note: for simplicity, we will assume from now until the end of the chapter that the message signal $m(t)$ used for PM modulation or its integration $\int_{-\infty}^{t} m(\alpha) d\alpha$ when $m(t)$ is used for FM modulation have magnitudes in the range of 1 volt. If this is not the case, the condition for having narrowband FM signal is $\left|k_f a(t)\right| \ll 1$ and having narrowband PM signals is $\left|k_p \cdot m(t)\right| \ll 1$.

To evaluate the bandwidth of this signal, we need to expand it using a power series expansion. So, we will define a slightly different signal

$$\hat{g}_{FM}(t) = A \cdot e^{j\{\omega_c t + k_f a(t)\}} = A \cdot e^{j\omega_c t} \cdot e^{jk_f a(t)}.$$

Remember that

$$\hat{g}_{FM}(t) = A \cdot e^{j\{\omega_c t + k_f a(t)\}} = A \cdot \cos\left[\omega_c t + k_f a(t)\right] + jA \cdot \sin\left[\omega_c t + k_f a(t)\right],$$

so

$$g_{FM}(t) = \mathrm{Re}\left\{\hat{g}_{FM}(t)\right\}.$$

Now we can expand the term $e^{jk_f a(t)}$ in $\hat{g}_{FM}(t)$, which gives

$$\begin{aligned}\hat{g}_{FM}(t) &= A \cdot e^{j\omega_c t} \cdot \left[1 + jk_f a(t) + \frac{j^2 k_f^2 a^2(t)}{2!} + \frac{j^3 k_f^3 a^3(t)}{3!} + \frac{j^4 k_f^4 a^4(t)}{4!} + \ldots\right] \\ &= A \cdot \left[e^{j\omega_c t} + jk_f a(t) e^{j\omega_c t} - \frac{k_f^2 a^2(t)}{2!} e^{j\omega_c t} - \frac{jk_f^3 a^3(t)}{3!} e^{j\omega_c t} + \frac{k_f^4 a^4(t)}{4!} e^{j\omega_c t} + \ldots\right]\end{aligned}$$

Since $k_f$ and $a(t)$ are real ($a(t)$ is real because it is the integral of a real function $m(t)$), and since $\mathrm{Re}\{e^{j\omega_c t}\} = \cos(\omega_c t)$ and $\mathrm{Re}\{je^{j\omega_c t}\} = -\sin(\omega_c t)$, then

$$\begin{aligned}g_{FM}(t) &= \mathrm{Re}\left\{\hat{g}_{FM}(t)\right\} \\ &= A \cdot \left[\cos(\omega_c t) - k_f a(t)\sin(\omega_c t) - \frac{k_f^2 a^2(t)}{2!}\cos(\omega_c t) + \frac{k_f^3 a^3(t)}{3!}\sin(\omega_c t) + \frac{k_f^4 a^4(t)}{4!}\cos(\omega_c t) + \ldots\right]\end{aligned}$$

The assumption we made for $k_f$ is that ($k_f << 1$). This assumption will result in making all the terms with powers of $k_f$ greater than 1 to be small compared to the first two terms. So, the following is a reasonable approximation for $g_{FM}(t)$

$$\boxed{g_{FM\,(Narrowband)}(t) \approx A \cdot \left[\cos(\omega_c t) - k_f a(t)\sin(\omega_c t)\right]}, \qquad \text{when } k_f << 1.$$

It must be stressed that the above approximation is only valid for narrowband FM signals that satisfy the condition ($k_f << 1$). The above signal is simply the addition (or actually the subtraction) of a cosine (the carrier) with a DSBSC signal (but using a sine as the carrier). The message signal that modulates the DSBSC signal is not $m(t)$ but its integration $a(t)$. One of the properties of the Fourier transform is that the bandwidth of a signal $m(t)$ and its integration $a(t)$ (and its derivative too) are the same (verify this). Therefore, the bandwidth of the narrowband FM signal is

$$\boxed{BW_{FM\ (Narrowband)} = BW_{DSBSC} = 2 \cdot BW_{m(t)} =}.$$

We will see later that when the condition ($k_f$ << 1) is not satisfied, the bandwidth of the FM signal becomes higher that twice the bandwidth of the message signal. Similar relationships hold for PM signals. That is

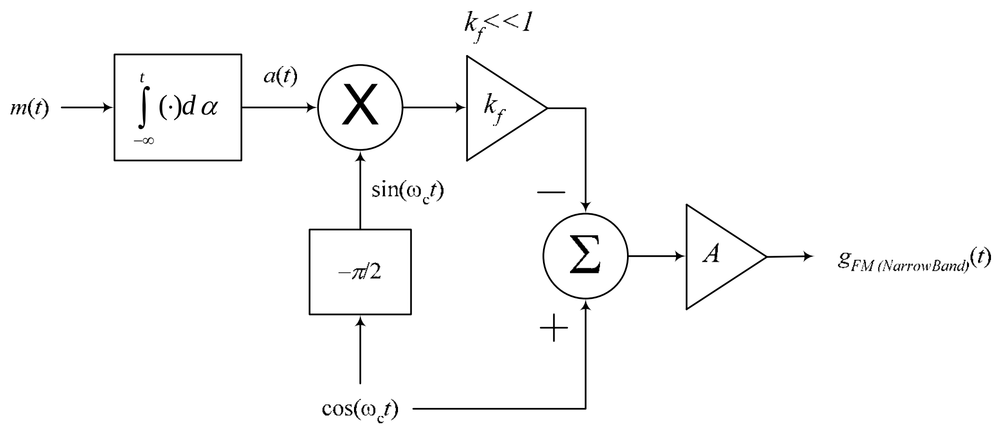
$$\boxed{g_{PM\ (Narrowband)}(t) \approx A \cdot \left[\cos(\omega_c t) - k_p m(t) \sin(\omega_c t)\right]}, \qquad \text{when } k_p << 1,$$

and

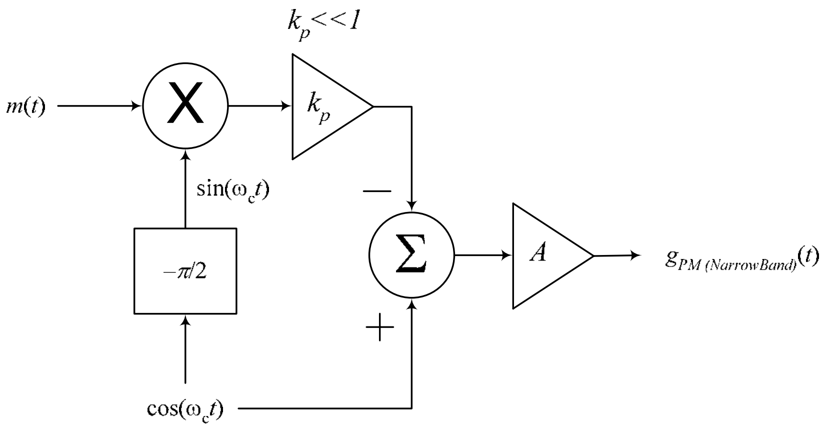
$$\boxed{BW_{PM\ (Narrowband)} = BW_{DSBSC} = 2 \cdot BW_{m(t)} =}.$$

## Construction of Narrowband Frequency and Phase Modulators

The above approximations for narrowband FM and PM can be easily used to construct modulators for both types of signals.




Narrowband FM Modulator




Narrowband PM Modulator