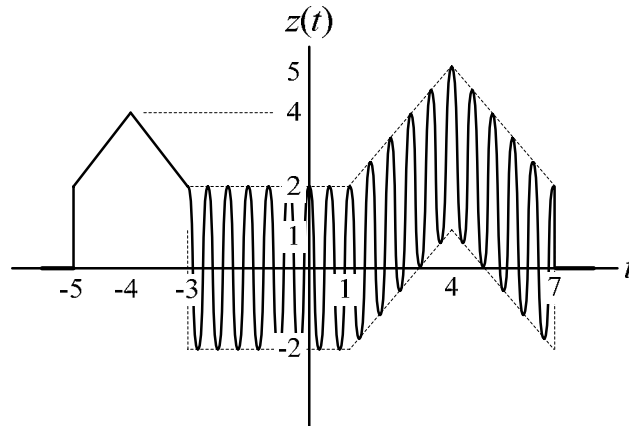
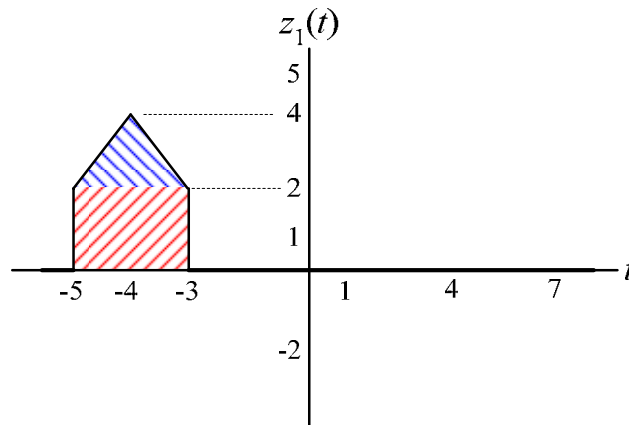


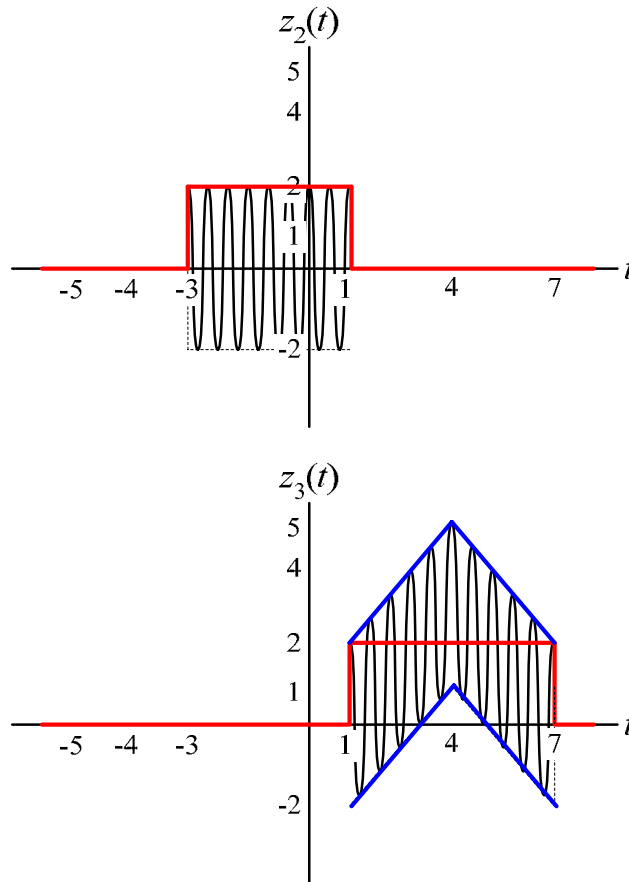
Example: Find the Fourier transform of the following signal $z(t)$ [assume the sinusoid has the shape of $\cos(6\pi t)$].



Solution: We need to decompose the signal $z(t)$ into simpler signals that will allow us to apply the FT to each component independently and then add the different FTs to get the overall FT of the signal $z(t)$.

We easily see that the signal $z(t)$ can be decomposed into the signals $z_1(t)$, $z_2(t)$, and $z_3(t)$ shown below





The signal $z_1(t)$ is itself composed of two signals: a Δ function with amplitude of 2, centered around $t = -4$, and has width of 2 seconds; and a rect function that also has amplitude of 2, is centered around $t = -4$, and has width of 2 seconds (note that the Δ function is NOT added to a constant of 2 since doing this will result in a constant of 2 everywhere with a triangular shape around $t = -4$, which is different from what we have here). Therefore,

$$z_1(t) = 2\Delta\left(\frac{t+4}{2}\right) + 2\text{rect}\left(\frac{t+4}{2}\right).$$

The signal $z_2(t)$ is basically a cosine function $[\cos(6\pi t)]$ as given in the problem] that is limited between $t = -3$ and $t = +1$. Limiting a signal can be achieved by multiplying the unlimited signal (the cosine function in this case which extends from $-\text{inf.}$ to $+\text{inf.}$) by a rect function that covers the range that we would like to limit the signal over. The center of the range we would like to limit the cosine over is $t = -1$ and the duration is 4 seconds. Therefore, we have to multiply the cosine function by a rect centered at $t = -1$ that has a width of 4 s. Therefore,

$$z_2(t) = 2\cos(6\pi t)\text{rect}\left(\frac{t+1}{4}\right).$$

The signal $z_3(t)$ is similar to $z_2(t)$ except that we have added a Δ function to it that is centered at $t = 4$ and has a width of 6 s and amplitude of 3. Note that the Δ function is added and not multiplied. This can be seen by the upper and lower covers of the cosine function (the blue lines) where both are moving in a parallel form (when one increases, the other one also increases and vice versa). Therefore,

$$z_3(t) = 2 \cos(6\pi t) \operatorname{rect}\left(\frac{t-4}{6}\right) + 3\Delta\left(\frac{t-4}{6}\right).$$

So,

$$\begin{aligned} z(t) &= z_1(t) + z_2(t) + z_3(t) \\ &= 2\Delta\left(\frac{t+4}{2}\right) + 2\operatorname{rect}\left(\frac{t+4}{2}\right) + 2 \cos(6\pi t) \operatorname{rect}\left(\frac{t+1}{4}\right) + 2 \cos(6\pi t) \operatorname{rect}\left(\frac{t-4}{6}\right) + 3\Delta\left(\frac{t-4}{6}\right). \end{aligned}$$

Using the linearity property of the FT,

$$Z(\omega) = Z_1(\omega) + Z_2(\omega) + Z_3(\omega).$$

Find the FT of each signal at a time gives

- Using FT 13 and 15 and FT property 7 in the table of the previous lecture.

$$\begin{aligned} Z_1(\omega) &= (2) \frac{2}{2} \operatorname{sinc}^2\left(\frac{2\omega}{4}\right) \cdot e^{-j\omega(-4)} + (2) 2 \operatorname{sinc}\left(\frac{2\omega}{2}\right) \cdot e^{-j\omega(-4)} \\ &= 2 \operatorname{sinc}^2\left(\frac{\omega}{2}\right) \cdot e^{j4\omega} + 4 \operatorname{sinc}(\omega) \cdot e^{j4\omega} \end{aligned}$$

- Using FT 13 and FT properties 7 and 13 in the table of the previous lecture.

$$\begin{aligned} Z_2(\omega) &= \frac{1}{2} \left[(2) 4 \operatorname{sinc}\left(\frac{4(\omega-6\pi)}{2}\right) \cdot e^{-j(\omega-6\pi)(-1)} + (2) 4 \operatorname{sinc}\left(\frac{4(\omega+6\pi)}{2}\right) \cdot e^{-j(\omega+6\pi)(-1)} \right] \\ &= 4 \operatorname{sinc}(2(\omega-6\pi)) \cdot e^{j(\omega-6\pi)} + 4 \operatorname{sinc}(2(\omega+6\pi)) \cdot e^{j(\omega+6\pi)} \end{aligned}$$

- Using FT 13 and 15 and FT properties 7 and 13 in the table of the previous lecture.

$$\begin{aligned}
Z_3(\omega) &= \frac{1}{2} \left[(2)6\text{sinc}\left(\frac{6(\omega-6\pi)}{2}\right) \cdot e^{-j(\omega-6\pi)(4)} + (2)6\text{sinc}\left(\frac{6(\omega+6\pi)}{2}\right) \cdot e^{-j(\omega+6\pi)(4)} \right] \\
&\quad + (3)\frac{6}{2}\text{sinc}^2\left(\frac{6\omega}{4}\right) e^{-j\omega(4)} \\
&= 6\text{sinc}(3(\omega-6\pi)) \cdot e^{-j4(\omega-6\pi)} + 6\text{sinc}(3(\omega+6\pi)) \cdot e^{-j4(\omega+6\pi)} + 9\text{sinc}^2\left(\frac{3\omega}{2}\right) e^{-j4\omega}
\end{aligned}$$

Now, just add the FTs given above to get the FT of $z(t)$.

Signal Transmission Through a Linear System

A communication system is usually described by its impulse response $h(t)$. The impulse response of a system is basically the output of that system when the input signal to that system is a unit impulse function $\delta(t)$. The impulse response of the system is the time-domain representation of that system. The FT of the impulse response denoted $H(\omega)$ is known as the frequency response of the system.

A signal $g(t)$ that is transmitted through the system with the impulse response $h(t)$ produces an output signal $y(t)$ that is given by the convolution equation

$$y(t) = g(t) * h(t).$$

In frequency domain, this can be represented as

$$Y(\omega) = G(\omega) \cdot H(\omega).$$

Decomposing this into a magnitude and a phase component gives

$$|Y(\omega)| e^{j\theta_Y(\omega)} = |G(\omega)| \cdot |H(\omega)| e^{j[\theta_G(\omega) + \theta_H(\omega)]}.$$

Distortionless Transmission

When transmitting a signal $g(t)$ through a communication system, the system may or may not distort the transmitted signal. A system that does not distort the transmitted signal is allowed to possibly change its magnitude and possibly delay it. If the output signal a specific communication system is an amplified/attenuated and delayed form of the input signal, than that system is called and distortionless communication system.

Therefore, the output of a distortionless communication system is

$$y(t) = kg(t - t_d),$$

where k is a constant, and t_d is a time delay that is greater than zero. In frequency domain, this gives

$$Y(\omega) = kG(\omega)e^{-j\omega t_d} \quad \rightarrow \quad H(\omega) = ke^{-j\omega t_d}$$

$$\rightarrow \quad |H(\omega)| = k \quad \& \quad \theta_H(\omega) = -\omega t_d$$

A system that is described by the above frequency response is known as a distortionless system or a linear phase system (the phase of the frequency response changes linearly with the frequency).

Notice that the impulse response of the system described by the frequency response $H(\omega)$ given above is

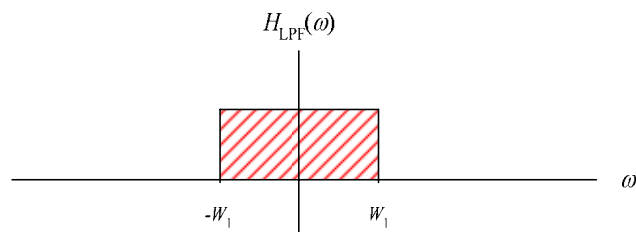
$$h(t) = k\delta(t - t_d).$$

Therefore, inputting an impulse function into this system produces a scaled and delayed impulse function at the output. The important thing here is that the input signal is not distorted but only delayed and scaled.

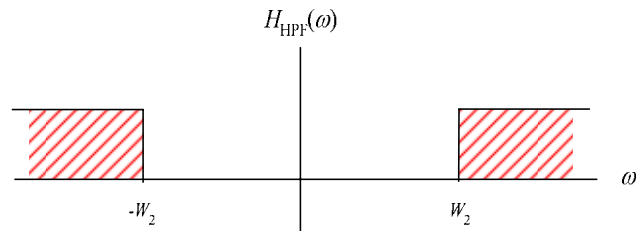
Electric Filters

Filters are electric devices that allow part of their input signals to pass and block part of their input signals. The distinction between the parts that are blocked and the parts that are allowed to pass is based on frequency. The range of frequencies that are allowed to pass is called the PASSBAND and the range of frequencies that are blocked is called STOPBAND. A Low-Pass Filter (LPF) is a filter that allows low frequencies up to a specified frequency to pass and block the rest of the frequencies. A High-Pass Filter, on the other hand, allows all frequency components that are above a specific frequency to pass and block the rest. A Band-Pass Filter is a filter that allows frequencies in a specific range that is greater than zero and less than infinity to pass and blocks frequencies above or below that range.

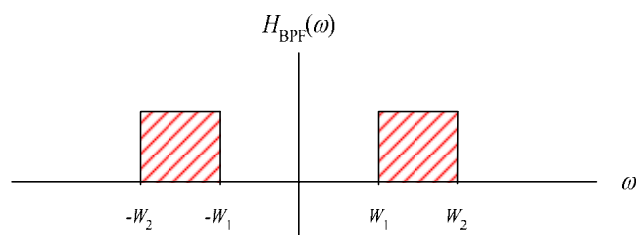
- a) **Low-Pass Filters (LPF):** a major characteristic of LPFs is the bandwidth of the filter. The bandwidth of a LPF is half the width of pulse of its frequency response (i.e., the width of the part of the pulse that is in the positive range of the frequency which is W_1). The frequency W_1 is also known as the CUTOFF frequency of the filter.



- b) **High-Pass Filters (HPF):** no bandwidth is defined for a HPF since the frequency response of that filter extends up to infinite. However, this filter is characterized by its CUTOFF frequency, which is W_1 as shown below.



- c) **Band-Pass Filters (BPF):** the BPF is characterized by two frequencies, W_1 known as the LOWER CUTOFF frequency, and W_2 known as the UPPER CUTOFF frequency. The bandwidth of that filter is also the width of pulse that is in the positive frequency region, or $BW = W_2 - W_1$.



Ideal vs. Real Filters:

The frequency responses for the three types of filters shown above are those of ideal filters. The reason is that there is an extremely sharp transition between the passbands and stopbands of these filters. The sharpness of the transition between passband and stopband is determined by something called the ORDER of the filter. The order of the filter is generally determined by the number of reactive components (capacitors and inductors) that are used in that filter. A zero-order filter (no capacitors or inductors) is basically a flat filter that allows all signals to pass. A first-order filter (one capacitor or inductor) is a filter that has very smooth transition between the passband and stopband. A second-order filter (number of reactive elements = number of capacitors + number of inductors = 2) has a sharper transition. The ideal filters shown above have in fact an infinite order (require an infinite number of inductors or capacitors, which makes them unrealizable (cannot be built in practice). Also an ideal filter would result in an infinite amount of delay between the input and output signals, which would make it useless even if you were able to build it.

