Relation between Fourier Series and Transform

The Fourier Transform (FT) is derived from the definition of the Fourier Series (FS). Consider, for example, the periodic complex signal $g_{To}(t)$ with period $T_0 = 2\pi/\omega_0$. The exponential FS of that signal allows the representation of $g_{To}(t)$ as

$$g_{To}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

Where the complex coefficients D_n are evaluated as

$$D_n = \frac{1}{T_0}\int_{T_0} g_{To}(t)e^{-jn\omega_0 t}dt$$

for any period T_0. For convenience, we will use the period $-T_0/2 \le t \le T_0/2$, which gives

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{To}(t)e^{-jn\omega_0 t}dt,$$

Note that the magnitudes and phases of the coefficients D_n are, respectively, known as the amplitude and phase spectrums of $g_{To}(t)$, or

$|D_n|$ is amplitude spectrum of $g_{To}(t)$, and
$\angle D_n$ is phase spectrum of $g_{To}(t)$.

Now, let us separate the different periods of $g_{To}(t)$ with zeros such that the different periods of $g_{To}(t)$ move away from each other (by holding one period of $g_{To}(t)$ and increasing the period duration T_0 until it becomes infinite). When $T_0 \to \infty$, → the signal $g_{To}(t)$ can be renamed $g(t)$ since it is no longer periodic but only contains one period of $g_{To}(t)$. In this case, $\omega_0 = 2\pi/T_0 \to 0$, and the discrete-time signal represented by the coefficients D_n vs. n becomes (in the limit) a continuous-time signal in terms of a new variable $\omega = n\omega_0$.

Thereofre,

$$D_n = \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t}dt$$

can be written as

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt,$$

where $\omega = n\omega_0$, and $D_n = G(n\omega_0)/T_0$.

Fourier Transform (FT) and Inverse Fourier Transform (IFT)

The FT of signal $g(t)$ is denoted $\mathcal{F}[g(t)]$ and is defined as

$$G(\omega) = \mathcal{F}[g(t)] = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt, \qquad (1)$$

and the IFT of $G(\omega)$ is denoted $\mathcal{F}^{-1}[G(\omega)]$ is defined as

$$g(t) = \mathcal{F}^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega \qquad (2)$$

We say that $g(t)$ and $G(\omega)$ for a FT pair, or $g(t) \Leftrightarrow G(\omega)$.

Notice that the exponent term in (1) has a negative sign but no negative sign exists in the exponent in (2). Also, notice that the integration in (1) is in terms of t and it is in terms of ω in (2).

Also notice that $G(\omega)$ in general is a complex signal that has both a magnitude $|G(\omega)|$ and a phase $\theta_G(\omega)$, or

$$G(\omega) = |G(\omega)| \cdot e^{j\theta_G(\omega)}.$$

General Properties of the FT

1) *Symmetry*

For **REAL** $g(t)$ ➔ $G(-\omega) = G^*(\omega)$

$|G(-\omega)| = |G^*(\omega)| = |G(\omega)|$

$\theta_G(-\omega) = -\theta_G(\omega)$

For REAL $g(t)$ with $g(-t) = g(t)$ (even functions) ➔ $G(\omega)$ is PURLEY REAL

For REAL $g(t)$ with $g(-t) = -g(t)$ (odd functions) ➔ $G(\omega)$ is PURELY IMAGINARY

2) *Existence of the FT*

For a signal $g(t)$, if $\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$, ➔ FT $\mathcal{F}[g(t)]$ exists.

The opposite is not necessarily true.
This above existence condition comes from the fact that

$$\int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \le \int_{-\infty}^{\infty} |\, g(t)e^{-j\omega t} \,|\, dt \le \int_{-\infty}^{\infty} |\, g(t)\,|\,.\,|\, e^{-j\omega t}\,|\, dt \le \int_{-\infty}^{\infty} |\, g(t)\,|\, dt < \infty$$

Therefore, if the integration of the magnitude of a function is finite, then the FT integral is also finite and therefore, the FT exists.

3) *Linearity*

If $g(t) \Leftrightarrow G(\omega)$ and $f(t) \Leftrightarrow F(\omega)$,

then

$$a\cdot g(t) + b\cdot f(t) \quad \Leftrightarrow \quad a\cdot G(\omega) + b\cdot F(\omega).$$

Meaning of Negative Frequency

(On page 57 of your textbook, the author explains the meaning of negative frequency. Here I'll rephrase what he mentioned in a slightly different way.)

We know that the frequency of any signal is always given in terms of a positive number. You, for example, would say that the frequency of a radio channel is 650 kHz and never say that it is – 650 kHz. So, what is the meaning of the part of the FT that falls to the left of the y–axis (the part with $\omega < 0$)?

It is known that you can describe the frequency spectrum of any real signal (such as all the signals that you can generate in the lab) using only one half of the frequency range (either positive or negative, but no need for both). So, what happens on the other side? The magnitude of the frequency response of any real function is an even function, and the phase of the frequency response is an odd function, so if we know either the negative or the positive halves, we can get the other half using these symmetry properties. For a purely imaginary signal, a similar thing happens and therefore, only one half of the frequency spectrum is needed. Now, a complex–valued signal is a combination of a real signal and an imaginary signal and therefore, it carries twice the amount of information as a real–valued or an imaginary–signal. So, a complex–valued signal would require twice the frequency range to describe its contents as a real– or imaginary–signal. This double the amount of information is basically described on the two halves of the spectrum. In fact, you can use only the positive half, but you would need two frequency spectrums to describe the frequency contents of that complex signal.

FT of Important Functions

1) FT of the Unit Impulse Function $\delta(t)$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega(0)} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\boxed{\delta(t) \Leftrightarrow 1}$$

2) FT of the Gate Function $\operatorname{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 0 & |t| < \tau/2 \\ 1/2 & |t| = \tau/2 \\ 1 & |t| > \tau/2 \end{cases}$

$$\mathcal{F}\left[\operatorname{rect}\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} = \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right)$$

$$= \frac{\tau}{\omega\tau/2}\left(\frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j}\right) = \frac{\tau \sin(\omega\tau/2)}{\omega\tau/2}$$

The function $\sin(x)/x$ is called $\operatorname{sinc}(x)$

$$\boxed{\operatorname{rect}(t/\tau) \Leftrightarrow \tau \cdot \operatorname{sinc}(\omega\tau/2)}$$

3) IFT of $\delta(\omega - \omega_0)$

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} dt = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\boxed{e^{j\omega_0 t} / 2\pi \Leftrightarrow \delta(\omega - \omega_0)}$$

4) FT of $\cos(\omega_0 t)$

$$\mathcal{F}[\cos(\omega_0 t)] = \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t} dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega_0 t} e^{-j\omega t} dt$$

$$= \frac{1}{2}\mathcal{F}\left[e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[e^{-j\omega_0 t}\right] = \frac{1}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

$$\boxed{\cos(\omega_0 t) \Leftrightarrow \pi\,[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}$$

Similarly,

$$\boxed{\sin(\omega_0 t) \Leftrightarrow (\pi/j)\,[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]}$$

What does Spectrum of a Signal Mean?

Now we know how to get the FT $G(\omega)$ from a signal $g(t)$ and how to get $g(t)$ back from $G(\omega)$ using the IFT formula. But, what does the FT of a signal physically mean? The FT of a signal represents the frequency contents of that signal. That is, what sines or cosines add up together to form the signal. So, it appears that the FT does the same thing as the FS. In fact, that is true. The difference between the FS and the FT is that the FS shows what sines and cosines with frequencies that are multiples of some fundamental frequency ω_0 combine to produce the PERIODIC signal. The periodicity of any signal that the FS simulates (even if the signal we are applying the FS to is not periodic, the FS automatically assumes the periodicity of the signal that is given in the period that we are integrating over) causes the spectrum of the signal to be a discrete–frequency signal that is defined only at multiples of ω_0. For general signals that are not periodic, the FT (which is a form of the FS) becomes a continuous–frequency signal that is defined for all values of ω. So, how much energy (or power) does a sine function, for example, with a specific frequency contribute to a signal that is not periodic? The answer is generally zero. Only if we take a range of frequencies, such as the range of 150 to 160 Hz, we can say that the contribution of the sine waves with frequencies in this range is 10 J (this comes from the fact that the integration of a signal with finite magnitude between the points $t = 5^-$ (just before 5) to $t = 5^+$ (just after 5) is always zero.

Properties of the Fourier Transform

If $\quad g(t) \Leftrightarrow G(\omega) \qquad$ ———— (1)

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \qquad \text{———— (2)}$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega \qquad \text{———— (3)}$$

a) Symmetry between the FT and IFT

Let s be a time variable and β be a frequency variable.

$$\mathcal{F}[G(t)] = H(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt$$

$$= 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(t)e^{jt(-\omega)}dt\right]$$

By comparing the term between the brackets with Equation (3) above, we get

$$\mathcal{F}[G(t)] = H(\omega) = 2\pi g(-\omega)$$

$$\boxed{G(t) \Leftrightarrow 2\pi g(-\omega)}$$

b) Time Scaling

$$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j\omega t} dt$$

if $a > 0$: Let $\tau = at$ ➔ $d\tau = a\, dt$ ➔ $\tau = -\infty \to \infty$

$$\mathcal{F}[g(at)] = \frac{1}{a}\int_{-\infty}^{\infty} g(at) e^{-j\left(\frac{1}{a}\omega\right)(at)} d(at) = \frac{1}{a}\int_{-\infty}^{\infty} g(\tau) e^{-j\left(\frac{1}{a}\omega\right)(\tau)} d\tau = \frac{1}{a} G\left(\frac{1}{a}\right)$$

if $a < 0$: Let $\tau = at$ ➔ $d\tau = a\, dt$ ➔ $\tau = \infty \to -\infty$

$$\mathcal{F}[g(at)] = \frac{1}{a}\int_{-\infty}^{\infty} g(at) e^{-j\left(\frac{1}{a}\omega\right)(at)} d(at) = \frac{1}{a}\int_{\infty}^{-\infty} g(\tau) e^{-j\left(\frac{1}{a}\omega\right)(\tau)} d\tau = -\frac{1}{a} G\left(\frac{1}{a}\right)$$

$$g(at) \Leftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$$

c) Time and Frequency Reversals

$$\mathcal{F}[g(-t)] = \int_{-\infty}^{\infty} g(-t) e^{-j\omega t} dt, \quad \text{Let } \tau = -t \quad ➔ \quad d\tau = -dt \quad ➔ \quad \tau = \infty \to -\infty$$

$$= -\int_{\infty}^{-\infty} g(\tau) e^{-j\omega(-\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) e^{-j(-\omega)\tau} d\tau = G(-\omega)$$

$\boxed{g(-t) \Leftrightarrow -G(-\omega)}$ (This can also be obtained using (b) above with $a = -1$)

d) Time Differentiation

$$\mathcal{F}\left[\frac{dg(t)}{dt}\right] = \mathcal{F}\left[\frac{d}{dt}\{g(t)\}\right]$$
$$= \mathcal{F}\left[\frac{d}{dt}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega\right\}\right]$$

Since the differentiation is with respect to t and the integration is with respect to ω, we can bring derivative inside the integral as

$$\mathcal{F}\left[\frac{dg(t)}{dt}\right] = \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\left\{G(\omega)e^{j\omega t}\right\}d\omega\right]$$
$$= \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega G(\omega)e^{j\omega t}d\omega\right]$$
$$= \mathcal{F}\left[\mathcal{F}^{-1}\{j\omega G(\omega)\}\right] = j\omega G(\omega)$$

$\boxed{dg(t)/dt \Leftrightarrow j\omega G(\omega)}$ and $\boxed{d^n g(t)/dt^n \Leftrightarrow (j\omega)^n G(\omega)}$

e) Time Integration

$$\mathcal{F}\left[\int_{-\infty}^{t} g(t)dt\right] = \mathcal{F}\left[\int_{-\infty}^{t} \{g(t)\}dt\right]$$
$$= \mathcal{F}\left[\int_{-\infty}^{t}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega\right\}dt\right]$$

Also here, the two integrals are in terms of different variables, so we can switch the order of integration as

$$\mathcal{F}\left[\int_{-\infty}^{t} g(\tau)d\tau\right] = \mathcal{F}\left[\int_{-\infty}^{t} \{g(\tau)\}d\tau\right]$$
$$= \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\{\int_{-\infty}^{t} e^{j\omega\tau}d\tau\right\}d\omega\right]$$

Since the period of integration in the inner integral is $-\infty \le \tau \le t$, we can insert a unit step function $u(t-\tau)$ that is equal to 1 in this region and 0 outside and change the limits of integration to be from $-\infty$ to ∞, which gives

$$\mathcal{F}\left[\int_{-\infty}^{t} g(\tau)d\tau\right] = \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\{\int_{-\infty}^{\infty} u(t-\tau)e^{j\omega\tau}d\tau\right\}d\omega\right].$$

By setting $s = -\tau$, we get

$$\mathcal{F}\left[\int_{-\infty}^{t} g(\tau)d\tau\right] = \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\{-\int_{\infty}^{-\infty} u(t+s)e^{-j\omega s}ds\right\}d\omega\right]$$
$$= \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\{\int_{-\infty}^{\infty} u(t+s)e^{-j\omega s}ds\right\}d\omega\right]$$

Notice that the inner integral is nothing but the FT of $u(t+s)$,

$$\mathcal{F}\left[\int_{-\infty}^{t} g(\tau)d\tau\right] = \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\{\left(\pi\delta(\omega)+\frac{1}{j\omega}\right)\right\}e^{-j\omega t}d\omega\right]$$
$$= \mathcal{F}\left[\frac{\pi}{2\pi}\int_{-\infty}^{\infty} G(\omega)\delta(\omega)e^{-j\omega t}d\omega\right] + \mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{j\omega}G(\omega)\right)e^{-j\omega t}d\omega\right]$$
$$= \pi\mathcal{F}\left[\mathcal{F}^{-1}\{G(0)\delta(\omega)\}\right] + \mathcal{F}\left[\mathcal{F}^{-1}\left\{\frac{1}{j\omega}G(\omega)\right\}\right]$$
$$= \pi G(0)\delta(\omega) + \frac{1}{j\omega}G(\omega)$$

$$\boxed{\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \pi G(0)\delta(\omega) + \frac{1}{j\omega}G(\omega)}$$

f) Time and Frequency Shifting

$\boxed{g(t-t_0) \Leftrightarrow G(\omega)\cdot e^{-j\omega t_0}}$ and $\boxed{g(t)\cdot e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)}$

g) Multiplying by a Sinusoid

Using the frequency shifting property in (e) above,

$\boxed{g(t)\cdot\cos(\omega_0 t) \Leftrightarrow (1/2)[G(\omega-\omega_0) + G(\omega+\omega_0)]}$

$\boxed{g(t)\cdot\sin(\omega_0 t) \Leftrightarrow (1/j2)[G(\omega-\omega_0) - G(\omega+\omega_0)]}$

h) Convolution of Two Signals

The convolution of two signals $g(t)$ and $f(t)$ is defined as

$$g(t) * f(t) = \int_{-\infty}^{\infty} g(\tau) * f(t-\tau)d\tau = \int_{-\infty}^{\infty} f(\tau) * g(t-\tau)d\tau$$

Similarly,

$$G(\omega)*F(\omega)=\int_{-\infty}^{\infty}G(s)*F(\omega-s)ds=\int_{-\infty}^{\infty}F(s)*G(\omega-s)ds$$

The FT of the convolution of two signals is the product of the two FTs, and the IFT of the convolution of two signals is the product of the two IFTs

$$\boxed{g(t)*f(t)\Leftrightarrow G(\omega)\cdot F(\omega)}\quad \text{and} \quad \boxed{g(t)f(t)\Leftrightarrow \frac{1}{2\pi}G(\omega)*F(\omega)}$$