# Integer Multiplication and Division

COE 308

Computer Architecture

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## Presentation Outline

- ❖ Unsigned Multiplication
- ❖ Signed Multiplication
- ❖ Faster Multiplication
- ❖ Unsigned Division
- ❖ Signed Division

## Unsigned Multiplication

- Paper and Pencil Example:

```
Multiplicand          1100₂ = 12
Multiplier       ×    1101₂ = 13
                 ---------------
                      1100
                     0000
                    1100
                   1100
                 ---------------
Product          10011100₂ = 156
```

> Binary multiplication is easy
> 0 × multiplicand = 0
> 1 × multiplicand = multiplicand

- m-bit multiplicand × n-bit multiplier = (m+n)-bit product
- Accomplished via shifting and addition
- Consumes more time and more chip area than addition



## Sequential Unsigned Multiplication

- Initialize Product = 0
- Check each bit of the Multiplier
- If Multiplier bit = 1 then Product = Product + Multiplicand
- Rather than shifting the multiplicand to the left

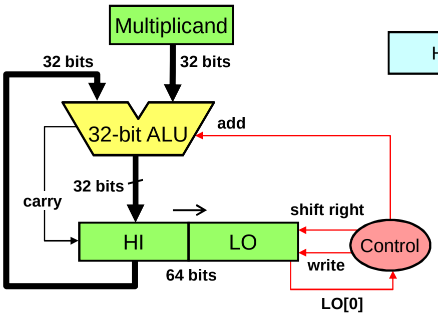
  Instead, Shift the Product to the Right

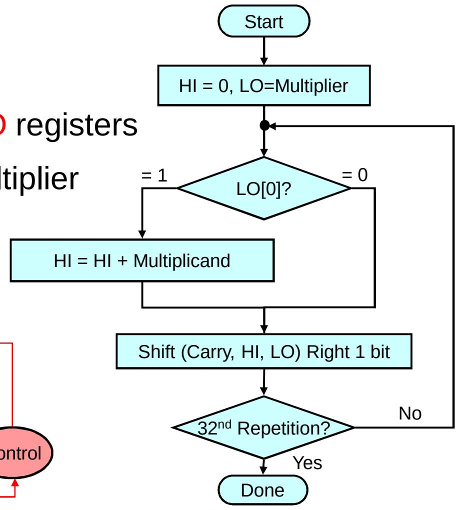
  Has the same net effect and produces the same result

  Minimizes the hardware resources
- One cycle per iteration (for each bit of the Multiplier)
  - Addition and shifting can be done simultaneously

# Sequential Multiplication Hardware

- Initialize HI = 0
- Initialize LO = Multiplier
- Final Product = HI and LO registers
- Repeat for each bit of Multiplier






# Sequential Multiplier Example

- Consider: $1100_2 \times 1101_2$ , Product = $10011100_2$
- 4-bit multiplicand and multiplier are used in this example
- 4-bit adder produces a 5-bit sum (with carry)

| Iteration | | Multiplicand | Carry | Product = HI, LO |
|---|---|---|---|---|
| 0 | Initialize (HI = 0, LO = Multiplier) | 1 1 0 0 | | 0 0 0 0 1 1 0 1 |
| 1 | LO[0] = 1 => ADD | | 0 | 1 1 0 0 1 1 0 1 |
| | Shift Right (Carry, HI, LO) by 1 bit | 1 1 0 0 | | 0 1 1 0 0 1 1 0 |
| 2 | LO[0] = 0 => Do Nothing | | | |
| | Shift Right (Carry, HI, LO) by 1 bit | 1 1 0 0 | | 0 0 1 1 0 0 1 1 |
| 3 | LO[0] = 1 => ADD | | 0 | 1 1 1 1 0 0 1 1 |
| | Shift Right (Carry, HI, LO) by 1 bit | 1 1 0 0 | | 0 1 1 1 1 0 0 1 |
| 4 | LO[0] = 1 => ADD | | 1 | 0 0 1 1 1 0 0 1 |
| | Shift Right (Carry, HI, LO) by 1 bit | 1 1 0 0 | | 1 0 0 1 1 1 0 0 |

# Next . . .

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- ❖ Signed Division



# Signed Multiplication

- ❖ So far, we have dealt with unsigned integer multiplication
- ❖ First Attempt:
  - ✧ Convert multiplier and multiplicand into positive numbers
    - ▪ If negative then obtain the 2's complement and remember the sign
  - ✧ Perform unsigned multiplication
  - ✧ Compute the sign of the product
  - ✧ If product sign < 0 then obtain the 2's complement of the product
- ❖ Better Version:
  - ✧ Use the unsigned multiplication hardware
  - ✧ When shifting right, extend the sign of the product
  - ✧ If multiplier is negative, the last step should be a subtract

# Signed Multiplication (Pencil & Paper)

❖ Case 1: Positive Multiplier

```
Multiplicand            1100₂ = -4
Multiplier          ×   0101₂ = +5
                    ---------------
Sign-extension      11111100
                    111100
                    ---------------
Product             11101100₂ = -20
```

❖ Case 2: Negative Multiplier

```
Multiplicand            1100₂ = -4
Multiplier          ×   1101₂ = -3
                    ---------------
Sign-extension      11111100
                    111100
                    00100     (2's complement of 1100)
                    ---------------
Product             00001100₂ = +12
```



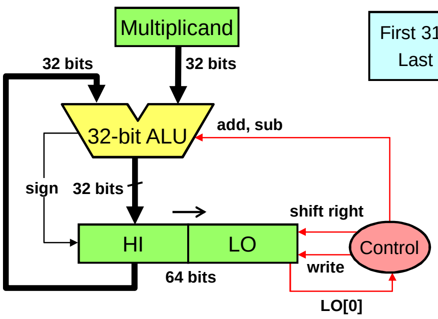
# Sequential Signed Multiplier

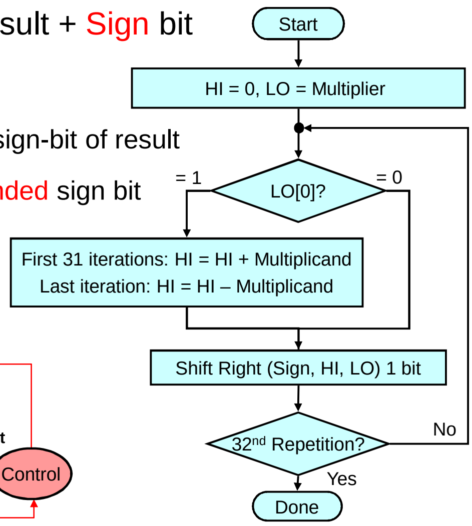
❖ ALU produces 32-bit result + Sign bit

❖ Check for overflow

✧ No overflow ➔ Extend sign-bit of result

✧ Overflow ➔ Invert extended sign bit



# Signed Multiplication Example

- Consider: $1100_2$ (-4) × $1101_2$ (-3), Product = $00001100_2$
- Check for overflow: No overflow ➔ Extend sign bit
- Last iteration: add 2's complement of Multiplicand

| Iteration | | Multiplicand | Sign | Product = HI, LO |
|---|---|---|---|---|
| 0 | Initialize (HI = 0, LO = Multiplier) | 1 1 0 0 | | 0 0 0 0 1 1 0 1 |
| 1 | LO[0] = 1 => ADD | | + 1 | 1 1 0 0 1 1 0 1 |
| | Shift (Sign, HI, LO) right 1 bit | 1 1 0 0 | | 1 1 1 0 0 1 1 0 |
| 2 | LO[0] = 0 => Do Nothing | | | |
| | Shift (Sign, HI, LO) right 1 bit | 1 1 0 0 | | 1 1 1 1 0 0 1 1 |
| 3 | LO[0] = 1 => ADD | | + 1 | 1 0 1 1 0 0 1 1 |
| | Shift (Sign, HI, LO) right 1 bit | 1 1 0 0 | | 1 1 0 1 1 0 0 1 |
| 4 | LO[0] = 1 => SUB (ADD 2's compl) | 0 1 0 0 | + 0 | 0 0 0 1 1 0 0 1 |
| | Shift (Sign, HI, LO) right 1 bit | | | 0 0 0 0 1 1 0 0 |



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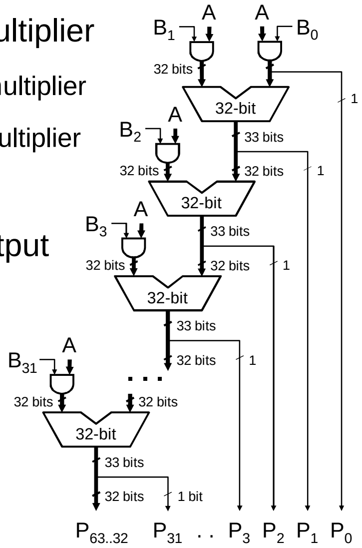
# Using Multiple Adders

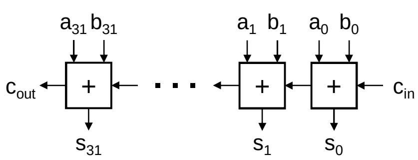
- ❖ 32-bit adder for each bit of the multiplier
  - ✧ 31 adders are needed for a 32-bit multiplier
  - ✧ AND multiplicand with each bit of multiplier
  - ✧ Product = accumulated shifted sum
- ❖ Each adder produces a 33-bit output
  - ✧ Most significant bit is a carry bit
  - ✧ Least significant bit is a product bit
  - ✧ Upper 32 bits go to next adder
- ❖ Array multiplier can be optimized
  - ✧ Carry save adders reduce delays



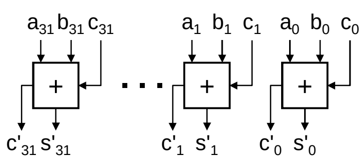



# Carry Save Adders

- ❖ Used when adding multiple numbers (as in multipliers)
- ❖ All the bits of a carry-save adder work in parallel
  - ✧ The carry does not propagate as in a carry-propagate adder
  - ✧ This is why a carry-save is faster than a carry-propagate adder
- ❖ A carry-save adder has 3 inputs and produces two outputs
  - ✧ It adds 3 numbers and produces partial sum and carry bits



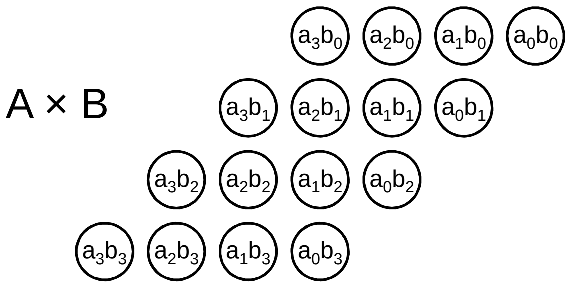

Carry-Propagate Adder




Carry-Save Adder

# Wallace Tree Multiplier - 1 of 2

- Suppose we want to multiply two numbers A and B
  - Example on 4-bit numbers: $A = a_3\ a_2\ a_1\ a_0$ and $B = b_3\ b_2\ b_1\ b_0$
- Step 1: AND (multiply) each bit of A with each bit of B
  - Requires $n^2$ AND gates and produces $n^2$ product bits
  - Position of $a_ib_j$ = (i+j). For example, Position of $a_2b_3$ = 2+3 = 5



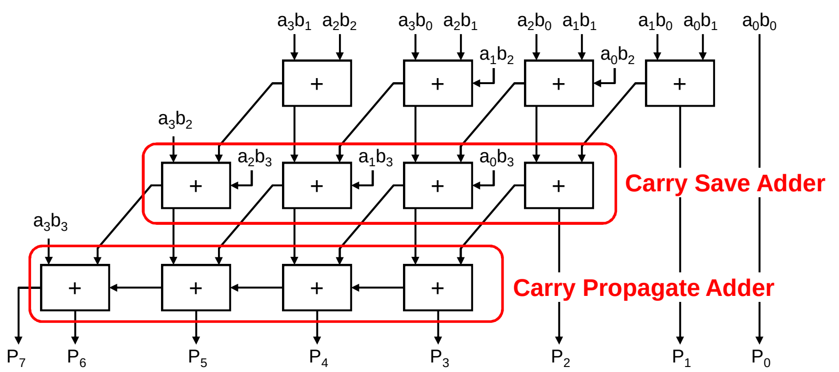



# Wallace Tree Multiplier – 2 of 2

Step 2: Use carry save adders to add the partial products

- Reduce the partial products to just two numbers

Step 3: Add last two numbers using a carry-propagate adder



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# Unsigned Division (Paper & Pencil)

```
                        10011₂ = 19        Quotient
Divisor    1011₂ ) 11011001₂ = 217        Dividend
                  -1011
                   ----
                     10
                     101
                     1010
                     10100
                     -1011
                     -----
                      1001
                      10011
                      -1011
                      -----
                       1000₂ = 8          Remainder
```

Try to see how big a number can be subtracted, creating a digit of the quotient on each attempt

Binary division is accomplished via shifting and subtraction

Dividend = Quotient × Divisor + Remainder

217 = 19 × 11 + 8

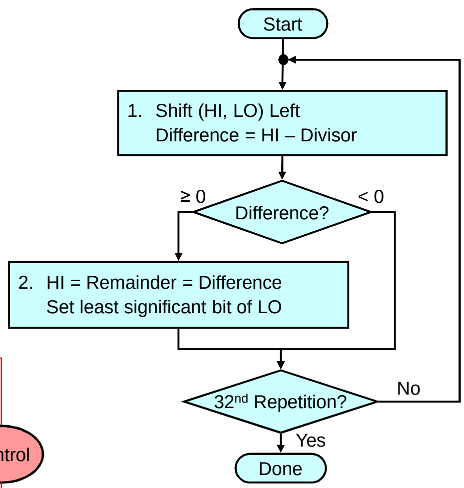
# Sequential Division

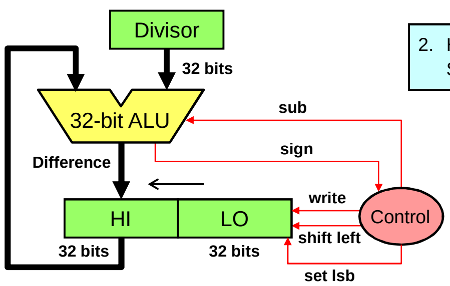
- ❖ Uses two registers: HI and LO
- ❖ Initialize: HI = Remainder = 0 and LO = Dividend
- ❖ Shift (HI, LO) LEFT by 1 bit (also Shift Quotient LEFT)
  - ✧ Shift the remainder and dividend registers together LEFT
  - ✧ Has the same net effect of shifting the divisor RIGHT
- ❖ Compute: Difference = Remainder – Divisor
- ❖ If (Difference ≥ 0) then
  - ✧ Remainder = Difference
  - ✧ Set Least significant Bit of Quotient
- ❖ Observation to Reduce Hardware:
  - ✧ LO register can be also used to store the computed Quotient



# Sequential Division Hardware

- ❖ Initialize:
  - ✧ HI = 0, LO = Dividend
- ❖ Results:
  - ✧ HI = Remainder
  - ✧ LO = Quotient



## Division Example

- Example: $1110_2$ / $0011_2$ (4-bit dividend & divisor)
- Result Quotient = $0100_2$ and Remainder = $0010_2$
- 4-bit registers for Remainder and Divisor (4-bit ALU)

| Iteration | | HI | | LO | Divisor | Difference |
|---|---|---|---|---|---|---|
| 0 | Initialize | 0 0 0 0 | | 1 1 1 0 | 0 0 1 1 | |
| 1 | 1: Shift Left, Diff = HI - Divisor | 0 0 0 1 | ← | 1 1 0 0 | 0 0 1 1 | 1 1 1 0 |
| | 2: Diff < 0 => Do Nothing | | | | | |
| 2 | 1: Shift Left, Diff = HI - Divisor | 0 0 1 1 | ← | 1 0 0 0 | 0 0 1 1 | 0 0 0 0 |
| | 2: Rem = Diff, set **lsb** of LO | 0 0 0 0 | | 1 0 0 1 | | |
| 3 | 1: Shift Left, Diff = HI - Divisor | 0 0 0 1 | ← | 0 0 1 0 | 0 0 1 1 | 1 1 1 0 |
| | 2: Diff < 0 => Do Nothing | | | | | |
| 4 | 1: Shift Left, Diff = HI - Divisor | 0 0 1 0 | ← | 0 1 0 0 | 0 0 1 1 | 1 1 1 1 |
| | 2: Diff < 0 => Do Nothing | | | | | |



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# Signed Division

- Simplest way is to remember the signs
- Convert the dividend and divisor to positive
  - Obtain the 2's complement if they are negative
- Do the unsigned division
- Compute the signs of the quotient and remainder
  - Quotient sign = Dividend sign XOR Divisor sign
  - Remainder sign = Dividend sign
- Negate the quotient and remainder if their sign is negative
  - Obtain the 2's complement to convert them to negative



# Signed Division Examples

1. **Positive** Dividend and **Positive** Divisor
   - Example: +17 / +3 Quotient = +5 Remainder = +2
2. **Positive** Dividend and **Negative** Divisor
   - Example: +17 / –3 Quotient = –5 Remainder = +2
3. **Negative** Dividend and **Positive** Divisor
   - Example: –17 / +3 Quotient = –5 Remainder = –2
4. **Negative** Dividend and **Negative** Divisor
   - Example: –17 / –3 Quotient = +5 Remainder = –2

The following equation must always hold:

**Dividend = Quotient × Divisor + Remainder**