Lecture 22

Tuesday, November 5, 2024 9:03 AM

Quantum Phase Estimation

- Recall eigenvectors & eigenvalues

$$Av = \lambda v \qquad v \text{ is called eigenvector of } A \text{ with Associated } \lambda \text{ eigenvalue}$$

Ex

$A = X$

eigenvector

$$X|+\rangle = 1|+\rangle$$

$$X|-\rangle = -1|-\rangle \qquad X|-\rangle = \frac{1}{\sqrt{2}}\left(\cancel{|0\rangle}^{|1\rangle} - \cancel{|1\rangle}^{|0\rangle}\right) = -1\left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right)$$

- Every unitary matrix has eigenvectors (eigenvalues)
- QPE prob.

Given a unitary matrix U and one of its eigenvectors $|v\rangle$, find or estimate its eigenvalue

Classically

$$U|v\rangle = \underline{e^{i\theta}}|v\rangle$$

$$\begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1N} \\ \vdots & \ddots & & \vdots \\ U_{N1} & \cdots & & U_{NN} \end{pmatrix}^{N\times N} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = e^{i\theta} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

$$U_{11}v_1 + U_{12}v_2 + \cdots + U_{1N}v_N = e^{i\theta}v_1$$

$$\vdots$$

N equations

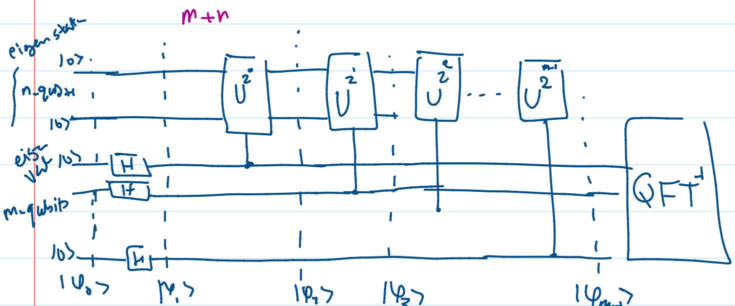
$$O(N) = O(2^n)$$

Quantum

eigenstate

$$\underbrace{\underbrace{|00\cdots0\rangle}_{m\text{-bit reg.}}\underbrace{|v\rangle}_{n\text{-qubits}}}_{}$$

$m+n$



$$|\psi_0\rangle = |00\cdots0\rangle|v\rangle$$

$$|\psi_1\rangle = |++\cdots\overset{U}{+}\rangle|v\rangle$$

$$|\psi_2\rangle = \cdots$$

$$|++\cdots+\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)|v\rangle$$ ← the rightmost qubit in the m-bit reg.

$$|\psi_2\rangle = |++\cdots\rangle \otimes \left(\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2i\theta}|1\rangle\right)\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)|v\rangle$$

$|++\cdots+\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right) |v\rangle$ ← the rightmost qubit in the n-bit reg.

$|\varphi_3\rangle = |++\cdots\rangle \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + e^{2i\theta}|1\rangle)\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right) |v\rangle$

$$|\varphi_{m-1}\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2^{m-1}i\theta}|1\rangle\right)\right) \cdots \left(\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)\right)|v\rangle$$

Let $\theta = 2\pi j \Rightarrow j = \frac{\theta}{2\pi}$ $\qquad j = j_{m-1} \cdots j_0$

$$|\varphi_{m-1}\rangle = \frac{1}{\sqrt{2^m}}\left(|0\rangle + e^{2\pi i j 2}|1\rangle\right) \cdots \left(|0\rangle + e^{2\pi i j}|1\rangle\right)|v\rangle$$

We know $0 \le \theta < 2\pi$, $0 \le j < 1$, $j = 0.j_1 \cdots j_m$

$$|\varphi_{m-1}\rangle = \frac{1}{\sqrt{2^m}}\left(|0\rangle + e^{2\pi i(0.j_m)}|1\rangle\right)\left(|0\rangle + e^{2\pi i(0.j_{m-1}j_m)}\right) \cdots \left(|0\rangle + e^{2\pi i(0.j_1 j_2 \cdots j_m)}\right)|v\rangle$$

$$QFT^{-1}|\varphi_{m-1}\rangle_{|v\rangle} = |\varphi_m\rangle_{|v\rangle} = \underbrace{|j_1 j_2 \cdots j_m\rangle}\,|v\rangle$$

$$|j\rangle \xrightarrow{QFT} \frac{1}{\sqrt{2^n}}\sum e^{2\pi i j k/N}|k\rangle$$

$$|j_1 j_2 \cdots j_m\rangle = e^{i\theta}$$

$$\frac{1}{\sqrt{2^n}}\sum e^{\cdots}|k\rangle \xrightarrow{QFT^{-1}} |j\rangle$$

$0 \le j < 1$

$$\frac{j_1}{2} + \frac{j_2}{4} + \cdots \frac{j_m}{2^n} = j$$

$$\theta = 2\pi j$$

It takes $O(m^2)$ qubits