Lecture 20

Thursday, October 31, 2024 8:43 AM

Ignored number to the left of decimal point

$$e^{2\pi i jk/N} = e^{2\pi i(0.j_0)k_{n-1}} e^{2\pi i(0.j_1 j_0)k_{n-2}} \dots \times e^{2\pi i(0.j_{n-2} \dots j_1 j_0)k_1} e^{2\pi i(0.j_{n-1} j_{n-2} \dots j_1 j_0)k_0}.$$

bring every together

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle$$
$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i(0.j_0)k_{n-1}} e^{2\pi i(0.j_1 j_0)k_{n-2}} \dots \times e^{2\pi i(0.j_{n-2} \dots j_1 j_0)k_1} e^{2\pi i(0.j_{n-1} j_{n-2} \dots j_1 j_0)k_0} |k\rangle.$$

Represent k a $k_{n-1} \dots k_1 k_0$ and unfolded the summation

$$\frac{1}{\sqrt{N}} \sum_{k_{n-1}=0}^{1} \dots \sum_{k_0=0}^{1} e^{2\pi i(0.j_0)k_{n-1}} e^{2\pi i(0.j_1 j_0)k_{n-2}} \dots \times e^{2\pi i(0.j_{n-2} \dots j_1 j_0)k_1} e^{2\pi i(0.j_{n-1} j_{n-2} \dots j_1 j_0)k_0} |k_{n-1} \dots k_0\rangle.$$

$|100011\rangle$

$$\frac{1}{\sqrt{N}} \sum_{k_{n-1}=0}^{1} \dots \sum_{k_0=0}^{1} e^{2\pi i(0.j_0)k_{n-1}} |k_{n-1}\rangle e^{2\pi i(0.j_1 j_0)k_{n-2}} |k_{n-2}\rangle \dots \times e^{2\pi i(0.j_{n-2} \dots j_1 j_0)k_1} |k_1\rangle e^{2\pi i(0.j_{n-1} j_{n-2} \dots j_1 j_0)k_0} |k_0\rangle.$$

$$\frac{1}{\sqrt{N}} \sum_{k_{n-1}=0}^{1} e^{2\pi i(0.j_0)k_{n-1}} |k_{n-1}\rangle \sum_{k_{n-2}=0}^{1} e^{2\pi i(0.j_1 j_0)k_{n-2}} |k_{n-2}\rangle \dots \times \sum_{k_1=0}^{1} e^{2\pi i(0.j_{n-2} \dots j_1 j_0)k_1} |k_1\rangle \sum_{k_0=0}^{1} e^{2\pi i(0.j_{n-1} j_{n-2} \dots j_1 j_0)k_0} |k_0\rangle.$$

Super Position

$N = 2^n$

$$\underbrace{\frac{1}{\sqrt{N}}}\left(|0\rangle + e^{2\pi i(0.j_0)}|1\rangle\right)\left(|0\rangle + e^{2\pi i(0.j_1 j_0)}|1\rangle\right)\cdots$$
$$\times\left(|0\rangle + e^{2\pi i(0.j_{n-2}\cdots j_1 j_0)}|1\rangle\right)\left(|0\rangle + e^{2\pi i(0.j_{n-1}j_{n-2}\cdots j_1 j_0)}|1\rangle\right).$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_0)}|1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_1 j_0)}|1\rangle\right)\cdots \qquad |j_{n-1}\rangle \qquad (7.9)$$
$$\times\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_{n-2}\cdots j_1 j_0)}|1\rangle\right)\underline{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_{n-1}j_{n-2}\cdots j_1 j_0)}|1\rangle\right)}.$$

Recall $\quad |j\rangle \longrightarrow \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i jk/N}|k\rangle$

Example

$N = 2 = 2^1 \longrightarrow n = 1$

$$|0\rangle \xrightarrow{QFT} \frac{1}{\sqrt{2}}\sum_{k=0}^{1} e^{2\pi i 0 k/N}|k\rangle = \frac{1}{\sqrt{2}}\sum_{k=0}^{1}|k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

10%:

1 bonus point

$$|1\rangle \xrightarrow{QFT} \frac{1}{\sqrt{2}}\sum_{k=0}^{1} e^{2\pi i k/2}|k\rangle = \frac{1}{\sqrt{2}}\left(e^{0}|0\rangle + e^{i\pi}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Homework $\quad N = 8 = 2^3 \rightarrow n = 3$

$$H|j_{n-1}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{j_{n-1}}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + (e^{i\pi})^{j_{n-1}}|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i j_{n-1}/2}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_{n-1})}|1\rangle\right).$$

We want to bring j_{n-2}, we construct a gate

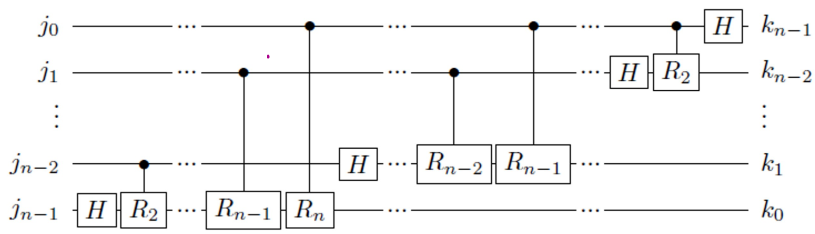
$$R_r|0\rangle = |0\rangle$$
$$R_r|1\rangle = \underline{e^{2\pi i/2^r}}\,|1\rangle$$
$$\Bigg\rangle \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^r} \end{pmatrix}$$

qubit

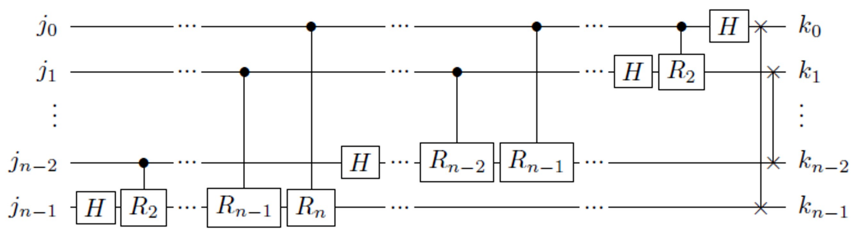
Controlled R gate

$\boxed{R_r}$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + \underline{e^{2\pi i(0.j_{n-1})}|1\rangle}\right) \xrightarrow{C\text{-}R_2} \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_{n-1})}\underline{(e^{2\pi i/2^2})^{j_{n-2}}}|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_{n-1})}e^{2\pi i(0.0j_{n-2})}|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_{n-1}j_{n-2})}|1\rangle\right).$$

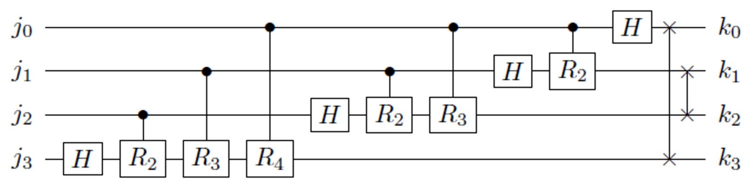


Note the order of the outputs is reversed, so we need to reverse the order, such as by using SWAP gates



This is our quantum circuit for the QFT. For example, with $n = 4$ qubits,

Computational basis

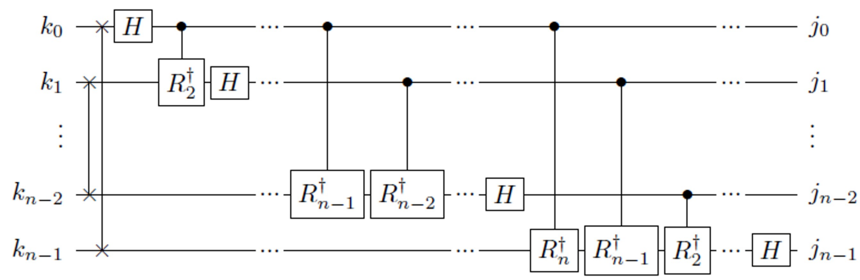


Fourier basis

time

frequency

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i jk/N}|k\rangle \longrightarrow |j\rangle$$



Tuesday

QPE – Quantum Phase Estimatn

$$U|v\rangle = e^{i\theta}|v\rangle$$

What is θ? $X \cdot |e^{i\theta}|^2$