King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 202 – Fundamentals of Computer Engineering

Term 081

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Number Systems – Base r

Number Systems – Base r

- General number in base r is written as:

$$A_{n-1}\, A_{n-2}\, \ldots\, A_2\, A_1\, A_0 \,.\, A_{-1}\, A_{-2}\, \ldots\, A_{-(m-1)}\, A_{-m}$$

Integer Part (n digits)

Radix Point

Fraction Part (m digits)

- Note that All A_i (digits) are less than r:
  - i.e. Allowed digits are 0, 1, 2, …, r – 1 ONLY
- A_{n-1} is the MOST SIGNIFACT Digit (MSD) of the number
- A_{-m} is the LEAST SIGNIFICANT Digit (LSD) of the number

A_{n-1} is the MSD of the integer part
A_0 is the LSD of the integer part
A_{-1} is the MSD of the fraction part
A_{-m} is the LSD of the fraction part

Number Systems – Base r

- The (base r) number

$$A_{n-1}\, A_{n-2}\, \ldots\, A_2\, A_1\, A_0 \,.\, A_{-1}\, A_{-2}\, \ldots\, A_{-(m-1)}\, A_{-m}$$

FORM or SHAPE OF NUMBER

is equal to

$$A_{n-1} \times r^{n-1} + A_{n-2} \times r^{n-2} + \ldots A_2 \times r^2 + A_1 \times r^1 + A_0 \times r^0 + A_{-1} \times r^{-1} + A_{-2} \times r^{-2} + \ldots A_{-(m-1)} \times r^{-(m-1)} + A_{-m} \times r^{-m}$$

VALUE OF NUMBER

Example – Decimal or Base 10

- For decimal system (base 10), the number

$$(724.5)_{10}$$

is equal to

$$7X10^{2} + 2X10^{1} + 4X10^{0} + 5X10^{-1}$$
$$= 7 X 100 + 2 X 10 + 4 X 1 + 5 X 0.1$$
$$= 700 + 20 + 4 + 0.5$$
$$= 724.5$$

It is all powers of 10:
…
$10^3 = 1000$,
$10^2 = 100$,
$10^1 = 10$,
$10^0 = 1$,
$10^{-1} = 0.1$,
$10^{-2} = 0.01$,
…

Example –Base 5

- Base 5 ➔ r = 5
- Allowed digits are: 0, 1, 2, 3, and 4 ONLY
- The number

$$(312.4)_5$$

is equal to

$$3X5^{2} + 1X5^{1} + 2X5^{0} + 4X5^{-1}$$
$$= 3 X 25 + 1 X 5 + 2 X 1 + 4 X 0.2$$
$$= 75 + 5 + 2 + 0.8$$
$$= (82.8)_{10}$$

Therefore $(312.4)_5 = (82.8)_{10}$

It is all powers of 5:
…
$5^3 = 125$,
$5^2 = 25$,
$5^1 = 5$,
$5^0 = 1$
$5^{-1} = 0.2$
$5^{-2} = 0.04$,
…

A Third Example –Base 2

- Base 2 ➔ r = 2
  - This is referred to as the **BINARY SYSTEM**
- Allowed digits are: 0 and 1 ONLY
- The number

$(110101.11)_2$

is equal to

$1X2^5 + 1X2^4 + 0X2^3 + 1X2^2 + 0X2^1 + 1X2^0 + 1X2^{-1} + 1X2^{-2}$

$= 1 X 32 + 1 X 16 + 1 X 4 + 1 X 2 + 1 X 0.5 + 1 X 0.25$

$= 32 + 16 + 4 + 1 + 0.5 + 0.25$

$= (53.75)_{10}$

Therefore $(110101.11)_2 = (53.75)_{10}$

It is all powers of 2:
…
$2^4 = 16$
$2^3 = 8$,
$2^2 = 4$,
$2^1 = 2$,
$2^0 = 1$
$2^{-1} = 0.5$
$2^{-2} = 0.25$,
…

Decimal to Binary Conversion of Integer Numbers

- Conversion from base 2 to base 10 (for real numbers) – See previous slide
- To convert a decimal *integer* to binary ➔ decompose into powers of 2
  - Example: $(37)_{10} = (?)_2$

37 has ONE 32 ➔ remainder is 5
5 has ZERO 16 ➔ remainder is 5
5 has ZERO 8 ➔ remainder is 5
5 has ONE 4 ➔ remainder is 1
1 has ZERO 2 ➔ remainder is 1
1 has ONE 1 ➔ remainder is 0

Therefore $(37)_{10} = (100101)_2$

Decimal to Binary Conversion of Integer Numbers– cont'd

- Or we can use the following (see table):
- You stop when the division result is ZERO
- Note the order of the resulting digits
- Therefore $(37)_{10}$ = $(100101)_2$
- To check:

$1X2^5+1X2+1 = 32+4+1 = 37$

No	No/2	Remainder	
37	18	1	← LSD
18	9	0	
9	4	1	
4	2	0	
2	1	0	
1	0	1	← MSD

In general: to convert a decimal integer to its equivalent in base r we use the above procedure but dividing by r

A Very Useful Table

- To represent decimal numbers from 0 till 15 (16 numbers) we need FOUR binary digits $B_3B_2B_1B_0$
- In general to represent N numbers, we need $\lceil \log_2 N \rceil$ bits
- Note than:
  - B_0 flipped or COMPLEMENTED at every increment
  - B_1 flipped or COMPLEMENTED every 2 steps
  - B_2 flipped or COMPLEMENTED every 4 steps
  - B_3 flipped or COMPLEMENTED every 8 steps

Decimal	Binary	Decimal	Binary
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111

A Very Useful Table – cont'd

- Note that zeros to the left of the number do not add to its value
- When we need DIGITS beyond 9, we will use the alphabets as shown in Table
  - Example: base 16 system has 16 digits; these are: 0, , 1, 2, 3, …, 8, 9, A, B, C, D, E, F
  - This is referred to as HEXADECIMAL or HEX number system

Decimal	Binary	Decimal	Binary
0	0000	8	1000
1	0001	9	1001
2	0010	10 → A	1010
3	0011	11 → B	1011
4	0100	12 → C	1100
5	0101	13 → D	1101
6	0110	14 → E	1110
7	0111	15 → F	1111

Decimal to Binary Conversion of Fractions

- Example: $(0.234375)_{10} = (?)_2$
- Solution: We use the following procedure
- **Note:**
  - The binary digits are the integer part of the multiplication process
  - The process stops when the number is 0
- There are situations where the process DOES NOT end – See next slide
- Therefore $(0.234375)_{10} = (0.001111)_2$
- To check: $(0.001111)_2 = 1X2^{-3} + 1X2^{-4} + 1X2^{-5} + 1X2^{-6} = (0.234375)_{10}$

No	NoX2	Integer Part	
0.234375	0.46875	0	← MSD
0.46875	0.9375	0	
0.9375	1.875	1	
0.875	1.75	1	
0.75	1.5	1	
0.5	1.0	1	← LSD
0			

In general: to convert a decimal fraction to its equivalent in base r we use the above procedure but multiplying by r

Decimal to Binary Conversion of Fractions – cont'd

- Example: $(0.513)_{10} = (?)_2$
- Solution: As in previous slide

No	NoX2	Integer Part
0.513	1.026	1
0.026	0.052	0
0.052	0.104	0
0.104	0.208	0
0.208	0.416	0
0.416	0.832	0
0.832	1.664	1
0.664	1.328	1
0.328	0.656	0
…		

Therefore $(0.513)_{10} = (0.100000110\ldots)_2$

If we chose to round to 1 significant figure ➔ $(0.1)_2$

Or to 7 significant figures ➔ $(0.1000001)_2$

Etc.

Octal Number System

- Base r = 8
- Allowed digits are = 0, 1, 2, …, 6, 7

- Example: the number $(127.4)_8$ has the decimal value $1X8^2 + 2X8^1 + 7X8^0 + 4X8^{-1}$

= 1 X 64 + 2 X 8 + 7 + 0.5

= $(87.5)_{10}$

It is all powers of 8:
…
$8^4 = 4096$
$8^3 = 512$,
$8^2 = 64$,
$8^1 = 8$,
$8^0 = 1$
$8^{-1} = 0.125$
$8^{-2} = 0.015625$,
…

Conversion between Octal and Binary

- **Example:** $(127)_8 = (?)_2$
- **Solution:** we can find the decimal equivalent (see previous slide) and then convert from decimal to binary

$(127)_8 = (87)_{10} \rightarrow (?)_2$

From long division

$(127)_8 = (87)_{10} = (1010111)_2$

To check:

$1X2^6+1X2^4+1X2^2+1X2^1+1X2^0$

$= 64 + 16 + 4 + 2 + 1$

$= 87$

No	No/2	Remainder
87	43	1
43	21	1
21	10	1
10	5	0
5	2	1
2	1	0
1	0	1

Conversion between Octal and Binary– cont'd

- **NOTE:** $(127)_8 = (1010111)_2$
- Lets group the binary digits in groups of 3 starting from the LSD

$(1010111)_2 \rightarrow (001 \quad 010 \quad 111)_2$

$\updownarrow \quad \updownarrow \quad \updownarrow$

1 2 7

- That is the decimal equivalent of the first group 111 → 7
  of the second group 010 → 2
  of the third group 001 → 1
- Hence, to convert from Octal to Binary one can perform direct translation of the Octal digits into binary digits: ONE Octal digit ←→ THREE Binary digits

Conversion between Octal and Binary – cont'd

- To convert from Binary to Octal, Binary digits are grouped into groups of three digits and then translated to Octal digits

- Example: $(1011101.10)_2 = (?)_8$
- Solution:

$$(1011101.10)_2 = (001\ 011\ 101\ .\ 100)_2$$
$$= (1 \quad 3 \quad 5 \quad . \quad 4)_8$$
$$= (135.4)_8$$

Note:
We can add zeros to the left of the number or to the right of the number after the radix point to form the groups

Conversion From Decimal to Octal

- **Problem**: What is the octal equivalent of $(\mathbf{32.57})_{10}$?
- **Solution**:

a) We can covert $(32.57)_{10}$ to binary and then to Octal or
b) We can do:

32_{10} ➔ 32/8 = 4 and remainder is 0 → 0
4/8 = 0 and remainder is 4 → 4

hence, $32_{10} = 40_8$

$(0.57)_{10}$ ➔ 0.57 X 8 = 4.56 → 4
0.56 X 8 = 4.48 → 4
0.48 X 8 = 3.84 → 3
0.84 X 8 = 6.72 → 6
…

hence, $(0.57)_{10} = (0.4436)_8$

What is $(0.4436)_8$ rounded for
-Two fraction digits?
-One fraction digit?

Therefore, $(32.57)_{10} = (40.4436)_8$

Hexadecimal Number Systems

- Base r = 16
- Allowed digits: 0, 1, 2, …, 8, 9, A, B, C, D, E, F
- The values for the alphabetic digits are as show in Table

Hex	Value
A	10
B	11
C	12
D	13
E	14
F	15

- **Example 1:**

$(B65F)_{16} = BX16^3+6X16^2+5X16^1+FX16^0$

$= 11X4096 + 6X256 + 5X16 + 15$

$= (46687)_{10}$

- **Example 2:**

$(1B.3C)_{16} = 1X16^1+BX16^0+3X16^{-1}+CX16^{-2}$

$= 16+11+3X0.0625+ 12X0.00390625$

$= (27.234375)_{10}$

Conversion Between Hex and Binary

- **Example:** $(1B.3C)_{16} = (?)_2$
- **Solution:** we can find the decimal equivalent (see previous slide) and then convert from decimal to binary

$(1B)_{16} = (27)_{10}$ ➔ $(?)_2$

From long division

$(1B)_{16} = (27)_{10} = (11011)_2$

$(0.3C)16 = (0.234375)_{10}=(0.001111)_2$

➔Therefore $(1B.3C)_{16} = (11011.\ 001111)_2$

Verify This Result

Conversion Between Hex and Binary – cont'd

- **Note:**

(1B.3C)$_{16}$ = (11011. 001111)$_2$ from previous example

Lets group the binary bits in groups of 4 starting from the radix point, adding zeros to the left of the number or to the right as needed

➔ (0001 1011 . 0011 1100)

⇕ ⇕ ⇕ ⇕

1 B . 3 C

- Hence, to convert from Hex to Binary one can perform direct translation of the Hex digits into binary digits: ONE Hex digit ←→ FOUR Binary digits

Conversion between Hex and Binary – cont'd

- To convert from Binary to Hex, Binary digits are grouped into groups of four digits and then translated to Hex digits

- Example: (1011101.10)$_2$ = (?)$_{16}$
- Solution:

(1011101.10)$_2$ = (0101 1101 . 1000)$_2$

= (5 D . 8)$_{16}$

= (5D.8)$_{16}$

Note:
We can add zeros to the left of the number or to the right of the number after the radix point to form the groups

Sample Exam Problem

- **Problem**: What is the radix r if

$$((33)_r + (24)_r) \text{ X } (10)_r = (1120)_r$$

- **Solution**:

$(33)_r = 3r + 3$,
$(24)_r = 2r + 4$,
$(10)_r = r$,
$(1120)_r = r^3 + r^2 + 2r$
therefore:

$$[(3r+3)+(2r+4)] \text{ X } r$$
$$= r^3 + r^2 + 2r \rightarrow r^3 - 4r^2 - 5r = 0, \text{ or}$$
$$r(r - 5)(r + 1) = 0$$

This means, the radix r is equal to 5

Number Ranges - Decimal

- Consider a decimal integer number of n digits:

$$A_{n-1}A_{n-2}\ldots A_1A_0 \quad \text{where } A_i \in \{0,1,2, \ldots, 9\}$$

Smallest integer is $0_{n-1}0_{n-2}\ldots 0_1 0_0 = 0$
Largest integer is $9_{n-1}9_{n-2}\ldots 9_1 9_0 = 10^n - 1$

Example: for n equal to 3 → 3 digits integer decimals; the maximum integer is 999 or $10^3 - 1$

Number Ranges – Decimal – cont'd

- Consider a decimal fraction of m digits:

  $0.A_{-1}A_{-2}\ldots A_{-(m-1)}A_{-m}$ where $A_i \in \{0,1,2, \ldots, 9\}$

Smallest non-zeros fraction is $0.0_{-1}0_{-2}\ldots 0_{-(m-1)}1_{-m} = 10^{-m}$
Largest fraction is $0.9_{-1}9_{-2}\ldots 9_{-(m-1)}9_{-m} = 1 - 10^{-m}$

Example: for m equal to 3 ➔ 3 digits decimal fraction;

The minimum fraction is 10^{-3} or 0.001
The maximum number is $1 - 10^{-3}$ or 0.999

Number Ranges – Base-r Numbers

- Consider a base-r integer of n digits:

  $A_{n-1}A_{n-2}\ldots A_1A_0$ where $A_i \in \{0,1,2, \ldots, r-1\}$

Smallest integer is $0_{n-1}0_{n-2}\ldots 0_1 0_0 = 0$
Largest integer is $(r-1)_{n-1}(r-1)_{n-2}\ldots(r-1)_1(r-1)_0 = r^n - 1$

Example: for r = 5, n equal to 3 ➔ 3 digits base-5 integer;
The maximum integer is $(444)_5$ or $(5^3 - 1)_{10}$
To check:
the decimal equivalent of $(444)_5$ is $4X5^2+4X5^1+4 = (124)_{10}$ or simply $5^3 - 1 = (124)_{10}$

Number Ranges - Base-r Numbers

- Consider a base-r fraction of m digits:

  $0.A_{-1}A_{-2}\ldots A_{-(m-1)}A_{-m}$ where $A_i \in \{0,1,2, \ldots, r-1\}$

Smallest non-zero fraction is

$$(0.0_{-1}0_{-2}\ldots 0_{-(m-1)}1_{-m})_r = (r^{-m})_{10}$$

Largest fraction is

$$(0.(r-1)_{-1}(r-1)_{-2}\ldots(r-1)_{-(m-1)}(r-1)_{-m})_r = (1 - r^{-m})_{10}$$

Example: for r = 5 and m equal to 3 ➔ 3 digits base-5 fraction;

The maximum number is $(0.444)_5$ or $1 - 5^{-3} = 0.992$

Number Ranges - Base-r Numbers – cont'd

		Decimal (r=10)	Binary (r = 2)	Octal (r = 8)	Hex (r = 16)
Integer	Min	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0
	Max	$9_{n-1}9_{n-2}\ldots9_19_0$ $= 10^n - 1$	$(1_{n-1}1_{n-2}\ldots1_11_0)_2$ $= (2^n - 1)_{10}$	$(8_{n-1}8_{n-2}\ldots8_18_0)_8$ $= (8^n - 1)_{10}$	$(F_{n-1}F_{n-2}\ldots F_1F_0)_{16}$ $= (16^n - 1)_{10}$
fraction	Min	$0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m}$ $= 10^{-m}$	$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_2$ $= (2^{-m})_{10}$	$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_8$ $= (8^{-m})_{10}$	$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_{16}$ $= (16^{-m})_{10}$
	Max	$0.9_{-1}9_{-2}\ldots9_{-(m-1)}9_{-m}$ $= 1 - 10^{-m}$	$(0.1_{-1}1_{-2}\ldots1_{-(m-1)}1_{-m})_2$ $= (1 - 2^{-m})_{10}$	$(0.7_{-1}7_{-2}\ldots7_{-(m-1)}7_{-m})_8$ $= (1 - 8^{-m})_{10}$	$(0.F_{-1}F_{-2}\ldots F_{-(m-1)}F_{-m})_{16}$ $= (1 - 16^{-m})_{10}$

Exercises

- What is 8^4 equal to in octal?
  $(8^4)_{10} = (10000)_8$
- What is 2^5 equal to in binary?
  $(2^5) = (100000)_2$
- What is $16^4 - 1$ equal to in Hex?
- What is $2^3 - 2^{-2}$ equal to in Binary?
- What is $16^5 - 16^4$ equal to in Hex?
- What is $3^4 - 3^{-2}$ equal to in base-3?
- What is $2^4 - 2^{-2}$ equal to in base-3?

Addition and Subtraction of (Unsigned) Numbers

Binary Addition of UNSIGEND Numbers

- Consider the following example:
  Find the summation of $(1100)_2$ and $(11001)_2$

Solution:

```
                    110000  ← Carry
  Augend             01100
  Addend            +11001
  ------------      ---------
  sum               100101
```

- Note that
  - 0+0 = 0, 0+1 = 1+0 = 1, and 1 + 1 = 0 and the carry is 1
  - If the maximum no of digits for the augend or the addend is n, then the summation has either n or n+1 digits
  - This procedure works even for non-integer binary numbers

Binary Subtraction of UNSIGEND Numbers

- Consider the following example:
  Subtract $(10010)_2$ from $(10110)_2$

Solution:

```
  Minuend            10110
  Subtrahend        -10010
  ------------      ---------
  Difference         00100
```

- Note that
  - $(10110)_2$ is greater than $(10010)_2$ → The result is POSITIVE
  - 0-0 = 0, 1-0 = 1, and 1 - 1 = 0
  - The difference size is always less or equal to the size of the minued or the subtrahend
  - This procedure works even for non-integer binary numbers

Binary Subtraction – cont'd

- Consider the following example:
  Subtract $(10011)_2$ from $(10110)_2$

Solution:

```
                     00110   ← Borrow
Minuen               10110
Subtrahend          -10011
------------        ---------
Difference          00011
```

- Note that
  - $(10110)_2$ is greater than $(10011)_2$ → result is positive
  - 0-1= 1, and the borrow from next significant digit is 1
  - This procedure works even for non-integer binary numbers

Binary Subtraction – cont'd

- Consider the following example:
  Subtract $(11110)_2$ from $(10011)_2$

Solution:

```
                                          00110   ← Borrow
Minuen               10011    \  /        11110
Subtrahend          -11110    /  \       -10011
------------        ---------  (2)      ----------  (1)
Difference          -01011   <---------   01011
```

- Note that
  - $(10011)_2$ is smaller than $(11110)_2$ → result is negative
  - This procedure works even for non-integer binary numbers

Binary Multiplication of UNSIGEND Numbers

- Consider the following example:
  Multiply $(1011)_2$ by $(101)_2$

Solution:

```
Multiplicand        1011
Multiplier         X 101
------------    ---------
                    1011
                   0000
                  1011
                 -------
Product          110111
```


Sums and Products in Base r (Unsigned Numbers)

- For sums and Products in base-r (r > 2) systems
  - Memorize tables for sums and products
  - Convert to Dec → perform operation → convert back to base-r
- **Example**: Find the summation of $(59F)_{16}$ and $(E46)_{16}$?

```
 010 ← Carry
 59F
+E46
------
13E5
```

1. F → 15
   6 → 6

   21_{10} → 16_{10} + 5_{10} → 15_H

2. 1
   9 → A → 10
   4 → 4 → 4

   14_{10} → E_H

3. 5 → 5
   E → 14

   19_{10} → 16_{10} + 3_{10} → 13_H

- This procedure is used for any base-r

Sums and Products in Base-r – cont'd

- **Example**: Find the multiplication of $(762)_8$ and $(45)_8$?
- **Solution**:

```
 3310  ← Carry (for 4)
 4310  ← Carry (for 5)
  762
 X 45
 ------
 4672
3710
-------
43772
```

Octal	Decimal	Octal
5X2	= 10 → 8 + 2	= 12
5X6+1	= 31 → 24 + 7	= 37
5X7+3	= 38 → 32 + 6	= 46
4X2	= 8 → 8 + 0	= 10
4X6+1	= 25 → 24 + 1	= 31
4X7+3	= 24+7	= 37

Therefore, product = $(43772)_8$

Decimal Codes

Decimal Codes

- There are 2^n **DISTINCT** n-bit binary codes (group of n bits)
  - n bits can count 2^n numbers
- For us, humans, it is more natural to deal with decimal digits rather than binary digits
- 10 different digits → we can use 4 bits to represent any digit
  - 3 bits count 8 numbers
  - 4 bits count 16 numbers → to represent 10 digits we need 4 bits at least

Binary Coded Decimal (BCD)

- Let the decimal digits be coded as show in table
- Then we can write numbers as

Decimal Digit	Binary Code	Decimal Digit	Binary Code
0	0000	5	0101
1	0001	6	0110
2	0010	7	0111
3	0011	8	1000
4	0100	9	1001

$(396)_{10}$ = $(0011\ 1001\ 0110)_{BCD}$

Since 3 → 0011, 9 = 1001, 6 = 0110

Although we are using the equal sign – but they are not equal in the mathematical sense; this is **JUST a code**

Note that $(396)_{10} = (110001100)_2 \neq (0011\ 1001\ 0110)_{BCD}$

BCD Addition – Example 1

- Consider:

000 ← Carry
241
+105

346

Addition in the Decimal Domain

```
        0010 → Carry
BCD for 1 = 0001
BCD for 5 = 0101
        --------
        0110 → BCD for 6

        0000 → Carry
BCD for 4 = 0100
BCD for 0 = 0000
        --------
        0100 → BCD for 4

        0000 → Carry
BCD for 2 = 0010
BCD for 1 = 0001
        --------
        0011 → BCD for 3
```

Addition in the Decimal Domain

Hence, we can add BCD codes to obtain the correct decimal result. Is true always?

BCD Addition – Example 2

- Consider:

110 ← Carry
448
+489

937

Addition in the Decimal Domain

```
        0010 → Carry
BCD for 8 = 1000
BCD for 9 = 1001
        --------
      1 0001 → > 9 → Need a correction step
       +0110    (add 6)
       --------
      1 0111 → (BCD for 7)

        0001 → Carry
BCD for 4 = 0100
BCD for 8 = 1000
        --------
        1101 → > 9 → Need a correction step
       +0110    (add 6)
       --------
      1 0011 → (BCD for 3)

        0001 → Carry
BCD for 4 = 0100
BCD for 4 = 0100
        --------
        1001 → BCD for 9
```

Addition in the Decimal Domain

BCD Subtraction – Example 3

- Consider:

110 ← Borrow
234
- 135

099

Subtraction in the Decimal Domain

```
            1 1111 → borrow
BCD for 4 =   0100
BCD for 5 =   0101
              --------
               1111 → a borrow occurred – need correction
             - 0110    (subtract 6)
              --------
               1001 → (BCD for 9 – Also Let A1 = A1 -1 = 2)

            1 1111 → borrow
BCD for 2 =   0010
BCD for 3 =   0011
              --------
               1111 → a borrow occurred – need correction
           -   0110 (subtract 6)
              --------
               1011 → (BCD for 9 – Also Let A2 = A2 -1 = 1)

             0  0000 → Carry
BCD for 1 =     0001
BCD for 1 =     0001
               --------
                0000 → BCD for 0 – no borrow occurred – no
                       need for correction
```

Subtraction in the Decimal Domain

BCD Addition – Summary

- BCD codes: decimal digits are assigned 4 bit codes
- We can perform additions using the BCD digits
  - If the result of adding two BCD digits is greater than 9, a correction step is required in order produce the correct BCD digit
  - To correct: add 6
  - If a carry is produced → move it to next BCD digits addition

Alphanumeric Codes

- We have
  - 10 decimal digits
  - 26 X 2 (English) letters: capital and small case
  - Some special characters {; , . : + - etc}
- If we assign each character of these a binary code, then computers can exchange alphanumeric information (letters, numbers, etc) by exchanging binary digits
- One binary code is the American Standard Code for Information Interchange (ASCII)

ASCII

- A 7-bits code ➔ 128 distinct codes
  - 96 printable characters (26 upper case letter, 26 lower case letters, 10 decimal digits, 34 non-alphanumeric characters)
  - 32 non-printable character
    - Formatting effectors (CR, BS, …)
    - Info separators (RS, FS, …)
    - Communication control (STX, ETX, …)
- Computers typically use words sizes that are multiples of 2
  - Usually 8 bits are used for the ASCII code with the 8^{th} (left most bit) bit set to zero, OR
  - The ASCII code is extended ➔Extended ASCII (platform dependant)
- A good reference about ASCII and Extended ASCII is found at http://www.cplusplus.com/doc/papers/ascii.html

ASCII – cont'd

- A 7-bits code ➔ 128 distinct codes
- The American Standard Code for Information Interchange (ASCII) uses seven binary digits to represent 128 characters as shown in the table.

```
00 NUL| 01 SOH| 02 STX| 03 ETX| 04 EOT| 05 ENQ| 06 ACK| 07 BEL
08 BS | 09 HT | 0A NL | 0B VT | 0C NP | 0D CR | 0E SO | 0F SI
10 DLE| 11 DC1| 12 DC2| 13 DC3| 14 DC4| 15 NAK| 16 SYN| 17 ETB
18 CAN| 19 EM | 1A SUB| 1B ESC| 1C FS | 1D GS | 1E RS | 1F US
20 SP | 21  ! | 22  " | 23  # | 24  $ | 25  % | 26  & | 27  '
28  ( | 29  ) | 2A  * | 2B  + | 2C  , | 2D  - | 2E  . | 2F  /
30  0 | 31  1 | 32  2 | 33  3 | 34  4 | 35  5 | 36  6 | 37  7
38  8 | 39  9 | 3A  : | 3B  ; | 3C  < | 3D  = | 3E  > | 3F  ?
40  @ | 41  A | 42  B | 43  C | 44  D | 45  E | 46  F | 47  G
48  H | 49  I | 4A  J | 4B  K | 4C  L | 4D  M | 4E  N | 4F  O
50  P | 51  Q | 52  R | 53  S | 54  T | 55  U | 56  V | 57  W
58  X | 59  Y | 5A  Z | 5B  [ | 5C  \ | 5D  ] | 5E  ^ | 5F  _
60  ` | 61  a | 62  b | 63  c | 64  d | 65  e | 66  f | 67  g
68  h | 69  i | 6A  j | 6B  k | 6C  l | 6D  m | 6E  n | 6F  o
70  p | 71  q | 72  r | 73  s | 74  t | 75  u | 76  v | 77  w
78  x | 79  y | 7A  z | 7B  { | 7C  | | 7D  } | 7E  ~ | 7F DEL
```


Unicode

- Unicode describes a 16-bit standard code for representing symbols and ideographs for the world's languages.

First 256 Codes for Unicode[a]

<table>
<tr><th></th><th colspan="2">Control</th><th colspan="6">ASCII</th><th colspan="2">Control</th><th colspan="6">Latin 1</th></tr>
<tr><th></th><th>000</th><th>001</th><th>002</th><th>003</th><th>004</th><th>005</th><th>006</th><th>007</th><th>008</th><th>009</th><th>00A</th><th>00B</th><th>00C</th><th>00D</th><th>00E</th><th>00F</th></tr>
<tr><td>0</td><td>CTRL</td><td>CTRL</td><td>SPACE</td><td>0</td><td>@</td><td>P</td><td>`</td><td>p</td><td>CTRL</td><td>CTRL</td><td>NB SP</td><td>°</td><td>À</td><td>Đ</td><td>à</td><td>D</td></tr>
<tr><td>1</td><td>CTRL</td><td>CTRL</td><td>!</td><td>1</td><td>A</td><td>Q</td><td>a</td><td>q</td><td>CTRL</td><td>CTRL</td><td>¡</td><td>±</td><td>Á</td><td>Ñ</td><td>á</td><td>ñ</td></tr>
<tr><td>2</td><td>CTRL</td><td>CTRL</td><td>"</td><td>2</td><td>B</td><td>R</td><td>b</td><td>r</td><td>CTRL</td><td>CTRL</td><td>¢</td><td>²</td><td>Â</td><td>Ò</td><td>â</td><td>ò</td></tr>
<tr><td>3</td><td>CTRL</td><td>CTRL</td><td>#</td><td>3</td><td>C</td><td>S</td><td>c</td><td>s</td><td>CTRL</td><td>CTRL</td><td>£</td><td>³</td><td>Ã</td><td>Ó</td><td>ã</td><td>ó</td></tr>
<tr><td>4</td><td>CTRL</td><td>CTRL</td><td>$|$</td><td>4</td><td>D</td><td>T</td><td>d</td><td>t</td><td>CTRL</td><td>CTRL</td><td>¤</td><td>´</td><td>Ä</td><td>Ô</td><td>ä</td><td>ô</td></tr>
<tr><td>5</td><td>CTRL</td><td>CTRL</td><td>%</td><td>5</td><td>E</td><td>U</td><td>e</td><td>u</td><td>CTRL</td><td>CTRL</td><td>¥|¥</td><td>µ</td><td>Å</td><td>Õ</td><td>å</td><td>õ</td></tr>
<tr><td>6</td><td>CTRL</td><td>CTRL</td><td>&</td><td>6</td><td>F</td><td>V</td><td>f</td><td>v</td><td>CTRL</td><td>CTRL</td><td>¦</td><td>¶</td><td>Æ</td><td>Ö</td><td>æ</td><td>ö</td></tr>
<tr><td>7</td><td>CTRL</td><td>CTRL</td><td>'</td><td>7</td><td>G</td><td>W</td><td>g</td><td>w</td><td>CTRL</td><td>CTRL</td><td>§</td><td>·</td><td>Ç</td><td>×</td><td>ç</td><td>÷</td></tr>
<tr><td>8</td><td>CTRL</td><td>CTRL</td><td>(</td><td>8</td><td>H</td><td>X</td><td>h</td><td>x</td><td>CTRL</td><td>CTRL</td><td>¨</td><td>¸</td><td>È</td><td>Ø</td><td>è</td><td>ø</td></tr>
<tr><td>9</td><td>CTRL</td><td>CTRL</td><td>)</td><td>9</td><td>I</td><td>Y</td><td>i</td><td>y</td><td>CTRL</td><td>CTRL</td><td>©</td><td>¹</td><td>É</td><td>Ù</td><td>é</td><td>ù</td></tr>
<tr><td>A</td><td>CTRL</td><td>CTRL</td><td>*</td><td>:</td><td>J</td><td>Z</td><td>j</td><td>z</td><td>CTRL</td><td>CTRL</td><td>ª</td><td>º</td><td>Ê</td><td>Ú</td><td>ê</td><td>ú</td></tr>
<tr><td>B</td><td>CTRL</td><td>CTRL</td><td>+</td><td>;</td><td>K</td><td>[</td><td>k</td><td>{</td><td>CTRL</td><td>CTRL</td><td>«</td><td>»</td><td>Ë</td><td>Û</td><td>ë</td><td>û</td></tr>
<tr><td>C</td><td>CTRL</td><td>CTRL</td><td>,</td><td><</td><td>L</td><td>\</td><td>l</td><td>|</td><td>CTRL</td><td>CTRL</td><td>¬</td><td>¼|1/4</td><td>Ì</td><td>Ü</td><td>ì</td><td>ü</td></tr>
<tr><td>D</td><td>CTRL</td><td>CTRL</td><td>-</td><td>=</td><td>M</td><td>]</td><td>m</td><td>}</td><td>CTRL</td><td>CTRL</td><td>-</td><td>½|1/2</td><td>Í</td><td>Ý</td><td>í</td><td>ý</td></tr>
<tr><td>E</td><td>CTRL</td><td>CTRL</td><td>.</td><td>></td><td>N</td><td>^</td><td>n</td><td>~|˜</td><td>CTRL</td><td>CTRL</td><td>®</td><td>¾|3/4</td><td>Î</td><td>þ</td><td>î</td><td>þ</td></tr>
<tr><td>F</td><td>CTRL</td><td>CTRL</td><td>/</td><td>?</td><td>O</td><td>_</td><td>o</td><td>CTRL</td><td>CTRL</td><td>CTRL</td><td>¯</td><td>¿</td><td>Ï</td><td>ß</td><td>ï</td><td>ÿ</td></tr>
</table>

[a] Unicode, Inc., The Unicode Standard: Worldwide Character Encoding, Version 1.0, Volume 1, © 1990, 1991 by Unicode, Inc. Reprinted by permission of Addison-Wesley Publishing Company, Inc.

Problems of Interest

- Problem List:

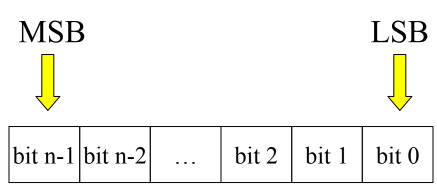
Signed Numbers Representations

Machine Representation of Numbers

- Computers store numbers in special digital electronic devices called REGISTERS
- REGISTERS consist of a fixed number of storage elements
- Each storage element can store one BIT of data (either 1 or 0)
- A register has a FINITE number of bits
  - Register size (n) is the number of bits in this register
  - N is typically a power of 2 (e.g. 8, 16, 32, 64, etc.)
  - A register of size n can represent 2^n distinct values
  - Numbers stored in a register can be either signed or unsigned

N-bit Register

- N-storage elements



- Each storage element capable of holding ONE bit (either 1 or –0
- n-bits can represent 2^n distinct values
  - For example if unsigned integer numbers are to be represented, we can represent all numbers from 0 to 2^n-1 (recall the number ranges for n-bits)
  - If we use it to represent signed numbers, still it can hold 2^n different numbers – we will learn about the ranges of these numbers in the coming slides

N-bit Register – cont'd

- Using a 4-bit register, $(13)_{10}$ or $(D)_H$ is represented as follows:

1	1	0	1

- Using an 8-bit register, $(13)_{10}$ or $(D)_H$ is represented as follows:

0	0	0	0	1	1	0	1

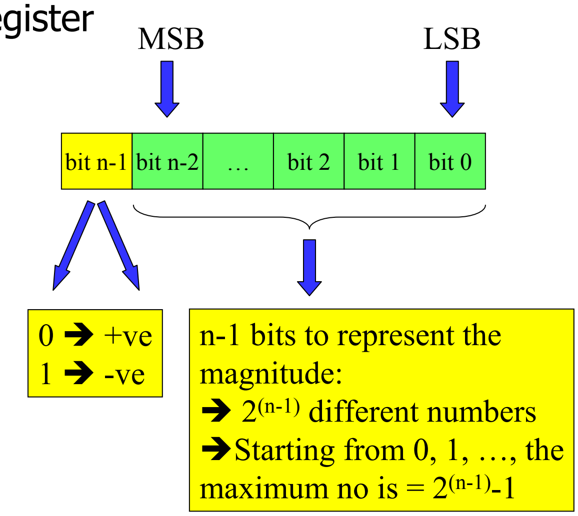
- Note that ZEROS are used to pad the binary representation of 13 in the 8-bit register
- We are still using UNSIGNED NUMBERS

Signed Number Representation

- To report a "signed" number, you need to specify its:
  - Magnitude (or absolute value), and
  - Sign (positive or negative)
- There are to main techniques to represent signed numbers
  1. Signed Magnitude Representation
  2. Complement Method

Signed Magnitude Representation

- N-bit register



Signed Magnitude Representation – Example 1:

- Show how +6, -6, +13, and –13 are represented using a 4-bit register
- Solution: Using a 4-bit register, the leftmost bit is reserved for the sign, which leaves 3 bits only to represent the magnitude

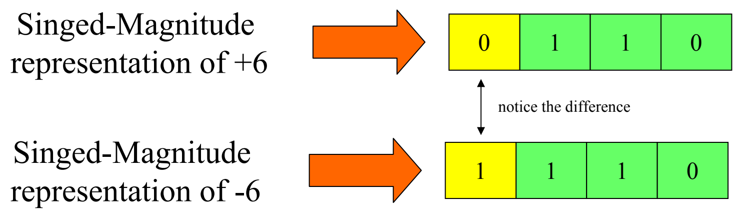
  ➔ The largest magnitude that can be represented = $2^{(4-1)} - 1 = 7 < 13$

  Hence, the numbers +13 and –13 can NOT be represented using the 4-bit register

Signed Magnitude Representation – Example 1: cont'd

- Solution (cont'd):

  However both –6 and +6 can be represented as follows:



Signed Magnitude Representation – Example 2:

- Show how +6, -6, +13, and –13 are represented using an 8-bit register

- Solution: Using an 8-bit register, the leftmost bit is reserved for the sign, which leaves 7 bits only to represent the magnitude

  ➔ The largest magnitude that can be represented = $2^{(8-1)} - 1 = 127$

  Hence, the numbers can be represented using the 8-bit register

Signed Magnitude Representation – Example 2: cont'd

- Solution (cont'd):

  Since 6 and 13 are equal to : 110 and 1101 respectively, the required representations are

Singed-Magnitude representation of +6	0	0	0	0	0	1	1	0
Singed-Magnitude representation of -6	1	0	0	0	0	1	1	0
Singed-Magnitude representation of +13	0	0	0	0	1	1	0	1
Singed-Magnitude representation of -13	1	0	0	0	1	1	0	1

Things We Learned About Signed-Magnitude Representation

- For an n-bit register
  - Leftmost bit is reserved for the sign (0 for +ve and 1 for –ve)
  - Remaining n-1 bits represent the magnitude
  - $2^{(n-1)}$ different numbers:
    - minimum is zero and maximum is $2^{(n-1)}-1$
- Two representations for zero: +0 and –0
- Range of numbers: from $-\{2^{(n-1)}-1\}$ to $+\{2^{(n-1)}-1\}$ ➔ symmetric range

Complement Representation

- +ve numbers (+N) are represented exactly the same way as in signed-magnitude representation
- -ve numbers (-N) are represented by the *complement* of N or N′

How is the complement of N or N′ defined?

N′ = M – N   where M is some constant

Properties of the Complement Representation

- The complement of the complement of N is equal to N:

Proof: (N′)′ = M – (M – N) = -(-N) = N

Same as with –ve numbers definition!

- The complement method representation of signed numbers simplifies implementation of arithmetic operations like subtraction:

e.g.: A – B can be replaced by A + (-B) or A + B′ using the complement method

Therefore to perform subtraction using computers we complement and add the subtrahend

How to Choose M?

- Consider the following number:

$$X = X_{n-1} \ldots X_2\, X_1\, X_0 \,\blacksquare\, X_{-1}\, X_{-2} \ldots X_{-(m-1)}\, X_{-m}$$

(n integral digits – m fractional digits)

- Using the base-r number system, there can be two types of the complement representation
  - Radix Complement (R's Complement)

  ➔ $M = r^n$
  - Diminished Radix Complement (R-1's Complement):

  ➔ $M = r^n - r^{-m}$

  $= r^n - \text{ulp}$

Recall that $r^n = 1_n 0_{n-1} \ldots 0_1 0_0$
= 1 followed by n zeros
Recall that $r^{-m} = 0\ldots00.00..01$
= unit in the least position

How to Choose M? – cont'd

- Note that:
  - $M = r^n - r^{-m}$ is the LARGEST unsigned number that can be represented
  - From the definitions of M, Rs complement of N is equal to R-1's complement of N plus ulp

Summary of Complement Method

- R's Complement:

Number System	R's Complement	Complement of X
Decimal	10's Complement	$X'_{10} = 10^n - X$
Binary	2's Complement	$X'_2 = 2^n - X$
Octal	8's Complement	$X'_8 = 8^n - X$
Hexadecimal	16's Complement	$X'_{16} = 16^n - X$

Summary of Complement Method – cont'd

- R-1's Complement:

Number System	R-1's Complement	Complement of X
Decimal	9's Complement	$X'_9 = (10^n - 10^{-m}) - X$ $= 99...99.99...99 - X$
Binary	1's Complement	$X'_1 = (2^n - 2^{-m}) - X$ $= 11...11.11...11 - X$
Octal	7's Complement	$X'_7 = (8^n - 8^{-m}) - X$ $= 77...77.77...77 - X$
Hexadecimal	15's Complement	$X'_{15} = (16^n - 16^{-m}) - X$ $= FF...FF.FF...FF - X$

Example 1a:

- Find the 9's and 10's complement of 2357?
- Solution:

$X = 2357 \Rightarrow n = 4$

$X'_9 = (10^4 - ulp) - X$

$= (10000 - 1) - 2357$

$= 9999 - 2357$

$= 7642$

$X'_{10} = 10^4 - X$

$= 10000 - 2357$

$= 7643$

Or alternatively,

$X'_{10} = X'_9 + ulp = 7642 + 1 = 7643$

Note that: $X + X'_9 = 2357 + 7642 = 9999 = M$

While $X + X'_{10} = 2357 + 7643 = 1\ 0000 = M$

Example 1b:

- Find the 9's and 10's complement of 2895.786?
- Solution:

$X = 2895.786 \Rightarrow n = 4, m = 3$

$X'_9 = (10^4 - ulp) - X$

$= (10000 - 0.001) - 2895.786$

$= 9999.999 - 2895.786$

$= 7104.213$

$X'_{10} = 10^4 - X$

$= 10000 - 2895.786$

$= 7104.214$

Or alternatively,

$X'_{10} = X'_9 + ulp = 7104.213 + 0.001 = 7104.214$

Note that: $X + X'_9 = 2895.786 + 7104.213 = 9999.999 = M$

While $X + X'_{10} = 2895.786 + 7104.214 = 1\ 0000.000 = M$

Example 2a:

- Find the 1's and 2's complement of 110101010?
- Solution:

X = 110101010 ➔ n = 9

$X'_1 = (2^9 – ulp) – X$

= (1000000000 –1) – 110101010

= 111111111 – 110101010

= 001010101

$X'_2 = 2^9 – X$

= 1000000000 – 110101010

= 001010110

Or alternatively,

$X'_2 = X'_1 + ulp$ = 001010101 + 1 = 001010110

> Note that: $X + X'_1$ = 110101010 + 001010101
> = 111111111 = M
> While $X + X'_2$ = 110101010 + 001010110
> = 1 000000000 = M

Example 2b:

- Find the 1's and 2's complement of 1010.001?
- Solution:

X = 1010.001 ➔ n = 4, m = 3

$X'_1 = (2^4 – ulp) – X$

= (10000 –0.001) – 1010.001

= 1111.111 – 1010.001

= 0101.110

$X'_2 = 2^4 – X$

= 10000 – 1010.001

= 0101.111

Or alternatively,

$X'_2 = X'_1 + ulp$ = 0101.110 + 0.001 = 0101.111

> Note that: $X + X'_1$ = 1010.001 + 0101.110
> = 1111.111 = M
> While $X + X'_2$ = 1010.001 + 0101.110
> = 1 0000.000 = M

Notes On 1's and 2's Complements Computation:

- 1's complement can be obtained by bitwise complementing the bits of X

Examples (from previous slide)

$X = 1010.001 \rightarrow X'_1 = 0101.110$

- 2's complement of X can be obtained by:
  1. Adding ulp to its 1's complement, or

     upl is added

     $X = 1010.001 \rightarrow X'_1 = 0101.110 \rightarrow X'_2 = 0101.111$

  2. Scanning X from right to left, copy all digits including first 1, complement all remaining digits

     $X = 1010.001 \rightarrow X'_2 = 0101.111$

     1[st] one you keep

     You complement subsequent bits

Example 3a:

- Find the 7's and the 8's complement of the following octal number 6770?
- Solution:

$X = 6770 \rightarrow n = 4$

$X'_7 = (8^4 - ulp) - X$

$= (10000 - 1) - 6770$

$= 7777 - 6770$

$= 1007$

$X'_8 = 8^4 - X$

$= 10000 - 6770$

$= 1010$

Or alternatively,

$X'_8 = X'_7 + ulp = 1007 + 1 = 1010$

Example 3b:

- Find the 7's and the 8's complement of the following octal number 541.736?
- Solution:

X = 541.736 ➔ n = 3, m = 3

$X'_7 = (8^3 - ulp) - X$

$= (10000 - 0.001) - 541.736$

$= 777.777 - 541.736$

$= 236.041$

$X'_8 = 8^3 - X$

$= 10000 - 541.736$

$= 236.042$

Or alternatively,

$X'_8 = X'_7 + ulp = 236.041 + 0.001 = 236.042$

Example 4a:

- Find the 15's and the 16's complement of the following Hex number 3FA9?
- Solution:

X = 3FA9 ➔ n = 4

$X'_{15} = (16^4 - ulp) - X$

$= (10000 - 1) - 3FA9$

$= FFFF - 3FA9$

$= C056$

$X'_{16} = 16^4 - X$

$= 10000 - 3FA9$

$= C057$

Or alternatively,

$X'_{16} = X'_{15} + ulp = C056 + 1 = C057$

Example 4b:

- Find the 15's and the 16's complement of the following Hex number 9B1.C70?
- Solution:

X = 9B1.C70 → n = 3, m = 3

$X'_{15} = (16^3 - ulp) - X$

= (1000 – 0.001) – 9B1.C70

= FFF.FFF – 9B1.C70

= 64E.38F

$X'_{16} = 16^3 - X$

= 1000 – 9B1.C70

= 64E.390

Or alternatively,

$X'_{16} = X'_{15} + ulp$ = 64E.38F + 0.001 = 64E.390

Complement Representation – Example 5:

- Show how +53 and -53 are represented in 8-bit registers using signed-magnitude, 1's complement and 2's complement?
- Solution:

Note that 53 = 32 + 16 + 4 + 1,

Therefore using 8-bit signed-magnitude:

- +53 → **0**0110101   -53 → **1**0110101

- To find the representation in complement method:

Complement Representation – Example 5: cont'd

- Solution: cont'd

To find the representation in complement method.
$(53)_{10} = (00110101)_2$ when written in 8-bit binary

1's complement ➔ 11001010 (inverting every bit)

2's complement ➔ 11001011 (adding ulp to X'_1)

Complement Representation – Example 5: cont'd

- Solution: cont'd

Putting all the results together in a table

	+53	-53
Signed-Magnitude	00110101	10110101
1's Complement	00110101	11001010
2's Complement	00110101	11001011

Note:
- +53 representation is the same for all methods
- For +53, the leftmost bit is 0 (+ve number)
- For -53, the leftmost bit is 1 (-ve number)

Example 6:

- For the shown 4-bit representations, indicate the corresponding decimal value in the shown representation

Example 6: cont'd

- Signed-Magnitude and 1's complement are symmetrical representations with TWO representations for ZERO
- Range from signed-magnitude and 1's complement is from -7 to +7
- 2's complement representation is not symmetrical
- Range for 2's complement is from -8 to +7 – with one representation for ZERO

	Unsigned	Signed-Magnitude	1's Complement	2's Complement
0000	0	0	0	0
0001	1	1	1	1
0010	2	2	2	2
0011	3	3	3	3
0100	4	4	4	4
0101	5	5	5	5
0110	6	6	6	6
0111	7	7	7	7
1000	8	-0	-7	-8
1001	9	-1	-6	-7
1010	10	-2	-5	-6
1011	11	-3	-4	-5
1100	12	-4	-3	-4
1101	13	-5	-2	-3
1110	14	-6	-1	-2
1111	15	-7	-0	-1

Summary

- The following table summarizes the properties and ranges for the studied signed number representations

	Signed-Magnitude	1's Complement	2's Complement
Symmetric	Y	Y	N
No of Zeros	2	2	1
Largest	$2^{(n-1)}-1$	$2^{(n-1)}-1$	$2^{(n-1)}-1$
Smallest	$-\{2^{(n-1)}-1\}$	$-\{2^{(n-1)}-1\}$	$-2^{(n-1)}$

Exercise

- Find the binary representation in signed magnitude, 1's complement, and 2's complement for the following decimal numbers: +13, -13, +39, +1, -1, +73, and -73. For all numbers, show the required representation for 6-bit and 8-bit registers

10's Complement

- For n = 1 and 2

X'_{10} (n=1)	X'_{10} using+/- in decimal
0	0
1	1
2	2
3	3
4	4
5	-5
6	-4
7	-3
8	-2
9	-1

X'_{10} (n=2)	X'_{10} using+/- in decimal
00	0
01	1
02	2
..	..
09	9
10	10
11	11
12	12
…	..
49	49
50	-50
51	-49
52	-48
…	…
98	-2
99	-1

8's Complement

- For n = 1 and 2

X'_8 (n=1)	X'_8 using+/- in decimal
0	0
1	1
2	2
3	3
4	-4
5	-3
6	-2
7	-1

X'_8 (n=2)	X'_8 using+/- in decimal
00	0
01	1
02	2
..	..
07	7
10	8
11	9
12	10
…	..
36	30
37	31
40	-32
41	-31
…	…
70	-8
71	-7
…	…
76	-2
77	-1

16's Complement

- For n = 1 and 2

X'_{16} (n=1)	X'_{16} using+/- in decimal
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	-8
9	-7
A	-6
B	-5
C	-4
D	-3
E	-2
F	-1

X'_{16} (n=2)	X'_{16} using+/- in decimal
00	0
01	1
…	…
0E	14
0F	15
10	16
11	17
…	…
1F	31
20	32
21	33
…	…
7E	126
7F	127
80	-128
81	-127
…	…
F0	-16
F1	-15
…	…
FD	-3
FE	-2
FF	-1

Operations On Binary Numbers

Operation On Binary Numbers

- Assuming we are dealing with n-bit binary numbers
  - UNSIGNED, or
  - SIGNED (2's complement)
- A subtraction can always be made into an addition operation A – B = A + (-B) or A + (B')
  - Compute the 2's complement of the subtrahend and added to the minuend

Operations on Binary Numbers

- The GENERAL OPERATION looks like:

$$
\begin{array}{rccccccl}
C_n & C_{n-1} & C_{n-2} & \ldots & C_2 & C_1 & C_0 & \leftarrow \text{Carry generated} \\
 & A_{n-1} & A_{n-2} & \ldots & A_2 & A_1 & A_0 & \rightarrow \text{Number A (signed or otherwise)} \\
+ & B_{n-1} & B_{n-2} & \ldots & B_2 & B_1 & B_0 & \rightarrow \text{Number B (signed or otherwise)} \\
\hline
C_n & S_{n-1} & S_{n-2} & \ldots & S_2 & S_1 & S_0 &
\end{array}
$$

- Note that although we start with n-bit numbers, we can end up with a result consisting of n+1 bits
  - Remember we are using n-bit registers!!
  - What to do with this extra bit (C_n)?

Addition of Unsigned Numbers - Review

- For n-bit words, the n-bit UNSIGNED binary numbers range from $(0_{n-1}0_{n-2}\ldots0_10_0)_2$ to $(1_{n-1}1_{n-2}\ldots1_11_0)_2$
  i.e. they range from 0 to 2^{n-1}
- When adding A to B as in:

```
Cn Cn-1 Cn-2 … C2 C1 C0     ← Carry generated
   An-1 An-2 … A2 A1 A0     → Number A (unsigned)
+  Bn-1 Bn-2 … B2 B1 B0     → Number B (unsigned)
-------------------
Cn Sn-1 Sn-2 … S2 S1 S0
```

- If C_n is equal to ZERO, then the result DOES fit into n-bit word $(S_{n-1}\ S_{n-2}\ \ldots\ S_2\ S_1\ S_0)$
- If C_n is equal to ONE, then the result DOES NOT fit into n-bit word → An "OVERFLOW" indicator!

Subtraction of Unsigned Numbers

- How to perform A – B (both defined as n-bit unsigned)?
- Procedure:
  1. Add the the 2's complement of B to A; this forms $A + (2^n - B)$
  2. If (A >= B), the sum produces end carry signal (C_n); discard this carry
  3. If A < B, the sum does not produce end carry signal (C_n); result is equal to $2^n - (B-A)$, the 2's complement of B-A – Perform correction:
     - Take 2's complement of sum
     - Place –ve sign in front of result
     - Final result is –(A-B)

Subtraction of Unsigned Numbers - NOTES

- Although we are dealing with unsigned numbers, we use the 2's complement to convert the subtraction into addition
- Since this is for UNSIGEND numbers, we have to use the –ve sign when the result of the operation is negative

Subtraction of Unsigned Numbers – Example (1)

- Example: X = 1010100 or $(84)_{10}$, Y = 1000011 or $(67)_{10}$ – Find X-Y and Y-X
- Solution:

n = 7

A) X – Y: X = 1010100
2's complement of Y = 0111101
Sum = 10010001
Discard C_n (last bit) = 0010001 or $(17)_{10}$ ← X – Y

B) Y – X: Y = 1000011
2's complement of X = 0101100
Sum = 1101111
C_n (last bit) is zero → need to perform correction
Y - X = -(2's complement of 1101111) = - 0010001

Subtraction of Unsigned Numbers – Example (2) – Base 10

- **Example: X = $(72532)_{10}$, Y = $(3250)_{10}$ – Find X-Y and Y-X**
- **Solution:**

A) X – Y: **X = 72532**

10's complement of Y = 96750

Sum = 169282

Discard C_n (last bit) = $(69282)_{10}$ ← X – Y

B) Y – X: **Y = 3250**

10's complement of X = 27468

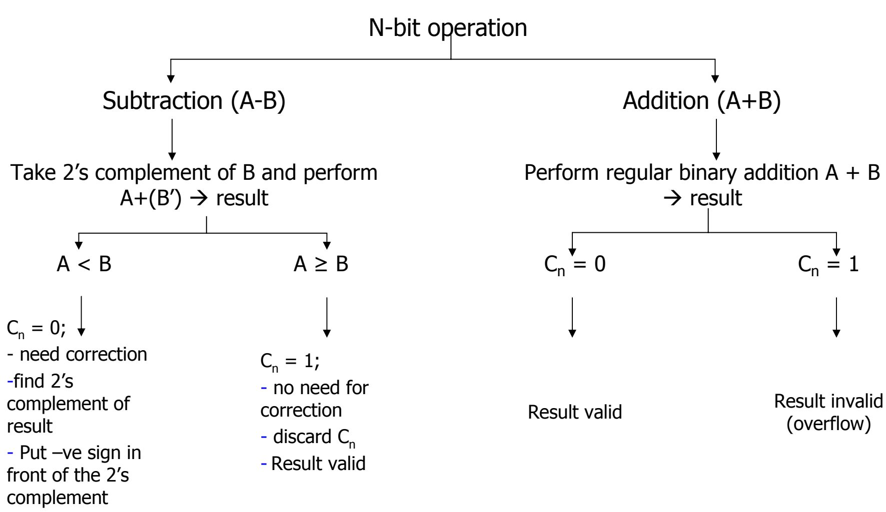
Sum = 30718

C_n (last bit) is zero → need to perform correction

Y - X = -(10's complement of 30718) = - 69282

The same procedure can be used for any base R system.

n-bit Unsigned Number Operations - Summary



2's Complement Review

- For n-bit words, the 2's complement SIGNED binary numbers range from $-(2^{n-1})$ to $+(2^{n-1}-1)$

  e.g. for 4-bit words, range = – 8 to +7
- Note that MSB is always 1 for –ve numbers, and 0 for +ve numbers

Addition/Subtraction of n-bit Signed Numbers by Example (1)

- Consider

```
       01 1000          111 0000
  +6   00 0110      -6   11 1010
+ 13   00 1101     +13   00 1101
------ -------------  -----  ------------
 +19   01 0011      +7   00 0111      Cn = 1 → discarded

       00 1100          110  0100
  +6   00 0110      -6   11 1010
-  13  11 0011     - 13  11 0011
------ -------------  -----  ------------
 - 7  11 1001      -19   101101       Cn = 1 → discarded
```

n = 6 → range -2^{6-1} = - 32 to $(2^{6-1}-1)$ = 31
Hence:
-All used numbers are valid (within the range)
-All results are also valid (within the range)

- Any carry out of sign bit position is DISCARDED
- -ve results are automatically in 2's complement form (no need for an explicit –ve sign)!

Are there cases when the results do not fit the n-bit register?

Addition/Subtraction of n-bit Signed Numbers by Example (2)

- **Consider**

C_n C_{n-1} carry	**0 10 0000**	C_n C_{n-1} carry	**1 10 0000**
16	**01 0000**	**-16**	**11 0000**
+ 23	**01 0111**	**+23**	**01 0111**
------	-------------	-----	------------
~~**+39**~~	~~**10 0011**~~	**+7**	**1 00 0111** (Result is valid, Discard C_n)

C_n C_{n-1} carry	**0 00 0000**	C_n C_{n-1} carry	**1 00 0000**
+16	**01 0000**	**-16**	**11 0000**
- 23	**10 1001**	**- 23**	**10 1001**
------	-------------	-----	------------
- 7	**11 1001** (Result is valid)	~~**-39**~~	~~**1 01 1001**~~

n = 6 ➔ range $-2^{6-1} = -32$ to $(2^{6-1}-1) = 31$

Hence:

-All used numbers are valid (within the range)

-All results are also valid (within the range)

✗ though we started with valid 6-bit signed numbers the results is in valid for a 6-bit signed representation

Addition/Subtraction of n-bit Signed Numbers by Example (2) – cont'd

- <u>NOTE:</u>
- The result is invalid (not within range) only if C_{n-1} and C_n are different! ➔ An OVERFLOW has occurred
- The result is valid (within range) if C_{n-1} and C_n are the same
  - If C_n = 1; it needs to be discarded
- If result is valid and –ve, it will be in the correct 2's complement form

Addition/Subtraction of n-bit Signed Numbers - Summary

- Summary