HW # 2 Solution

Q1. $f = \sum m(0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15)$

(i) off-set :

$$f = \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + \bar{a}b\bar{c}\bar{d} + \bar{a}b\bar{c}d$$
$$+ \bar{a}bc\bar{d} + \bar{a}bcd + a\bar{b}\bar{c}\bar{d} + a\bar{b}\bar{c}d + a\bar{b}c\bar{d}$$
$$+ ab\bar{c}d + abc\bar{d} + abcd$$

$$= \bar{a}\,[\bar{b}\bar{c}\bar{d} + \bar{b}\bar{c}d + \bar{b}c\bar{d} + b\bar{c}\bar{d} + b\bar{c}d + bc\bar{d} + bcd]$$
$$+ a\,[\bar{b}\bar{c}\bar{d} + \bar{b}\bar{c}d + \bar{b}c\bar{d} + b\bar{c}d + bc\bar{d} + bcd]$$

$$= \bar{a}\,[\bar{b}\,[\bar{c}\bar{d} + \bar{c}d + c\bar{d}] + b\,[\bar{c}\bar{d} + \bar{c}d + c\bar{d} + cd]]$$
$$+ a\,[\bar{b}\,[\bar{c}\bar{d} + \bar{c}d + c\bar{d}] + b\,[\bar{c}d + c\bar{d} + cd]]$$

$$= \bar{a}\,[\bar{b}\,[\bar{c}\,[\bar{d}+d] + c\,[\bar{d}]]$$
$$+ b\,[\bar{c}\,[\bar{d}+d] + c\,[\bar{d}+d]]]$$
$$+ a\,[\bar{b}\,[\bar{c}\,[\bar{d}+d] + c\,[\bar{d}]]$$
$$+ b\,[\bar{c}\,[d] + c\,[\bar{d}+d]]]$$

$$\Rightarrow \bar{f} = \bar{a}\,[\bar{b}\,[\bar{c}\,[0] + c[d]]$$
$$+ b\,[\bar{c}\,[0] + c\,[0]]]$$
$$+ a\,[\bar{b}\,[\bar{c}\,[0] + c\,[d]]$$
$$+ b\,[\bar{c}\,[\bar{d}] + c\,[0]]]$$

$$= \bar{a}\bar{b}cd + a\bar{b}cd + ab\bar{c}\bar{d}$$

(ii) Prime Implicants :

Based on the expansion given in part (i)

P.I.'s of $f_{\bar{a}\bar{b}} = scc\{\bar{c}, c\bar{d}, \bar{d}\} = \{\bar{c}, \bar{d}\}$

P.I.'s of $f_{\bar{a}b} = scc\{\bar{c}, c, 1\} = \{1\}$

P.I.'s of $f_{\bar{a}} = scc\{\bar{b}\bar{c}, \bar{b}\bar{d}, b, \bar{c}, \bar{d}\} = \{b, \bar{c}, \bar{d}\}$

P.I.'s of $f_{a\bar{b}} = scc\{\bar{c}, c\bar{d}, \bar{d}\} = \{\bar{c}, \bar{d}\}$

P.I.'s of $f_{ab} = scc\{\bar{c}d, c, d\} = \{c, d\}$

P.I.'s of $f_{a} = \{\bar{b}\bar{c}, \bar{b}\bar{d}, bc, bd; c\bar{d}, \bar{c}d\}$

$= \{\bar{b}\bar{c}, \bar{b}\bar{d}, bc, bd, c\bar{d}, \bar{c}d\}$

Thus, the P.I.'s of f =

$scc\{\bar{a}b, \bar{a}\bar{c}, \bar{a}\bar{d}, a\bar{b}\bar{c}, a\bar{b}\bar{d}, abc, abd, ac\bar{d}, a\bar{c}d, \bar{b}\bar{c}, \bar{b}\bar{c}\bar{d}, \bar{b}c\bar{d}, \bar{b}\bar{d}, bc, bc\bar{d}, bd, b\bar{c}d, bc\bar{d}, c\bar{d}, b\bar{c}d, \bar{c}d\}$

$= \{\bar{a}b, \bar{a}\bar{c}, \bar{a}\bar{d}, \bar{b}\bar{c}, \bar{b}\bar{d}, bc, bd, c\bar{d}, \bar{c}d\}$

Thus, there are 9 prime implicants in this function.

(iii) Expand Procedure

First, we represent the cubes in positional cube notation and compute the weight of each implicant

	a	b	c	d	weight
$\bar{a}\bar{b}\bar{c}\bar{d}$	10	10	10	10	27
$\bar{a}\bar{b}\bar{c}d$	10	10	10	01	26
$\bar{a}\bar{b}c\bar{d}$	10	10	01	10	26
$\bar{a}b\bar{c}\bar{d}$	10	01	10	10	28
$\bar{a}b\bar{c}d$	10	01	10	01	27
$\bar{a}bc\bar{d}$	10	01	01	10	27
$\bar{a}bcd$	10	01	01	01	26
$a\bar{b}\bar{c}\bar{d}$	01	10	10	10	26
$a\bar{b}\bar{c}d$	01	10	10	01	25
$a\bar{b}c\bar{d}$	01	10	01	10	25
$ab\bar{c}d$	01	01	10	01	26
$abc\bar{d}$	01	01	01	10	26
$abcd$	01	01	01	01	25
	76	67	76	76	

We first expand the implicant with the minimum weight. We have three implicants with the same weight $a\bar{b}\bar{c}d$, $a\bar{b}c\bar{d}$, and $abcd$.

Let us assume that we expand first the cube $a\bar{b}\bar{c}d$. Free set = $\{1, 4, 6, 7\}$.

No column can always be raised.

Intersecting with the off-set, one can see column 6 can't be raised. So, the remaining free set is $\{1,4,7\}$.

overexpandcube is $\bar{c}$.

We only need to check implicants $\bar{a}\bar{b}\bar{c}d$, $\bar{a}b\bar{c}d$, $ab\bar{c}d$, $a\bar{b}\bar{c}\bar{d}$, $\bar{a}\bar{b}\bar{c}d$, $\bar{a}b\bar{c}\bar{d}$ for being feasibly covered with the cube $a\bar{b}\bar{c}d$.

supercube $(a\bar{b}\bar{c}d,\ \bar{a}\bar{b}\bar{c}\bar{d}) = \bar{b}\bar{c}$ feasible
supercube $(a\bar{b}\bar{c}d,\ \bar{a}\bar{b}\bar{c}d) = \bar{b}\bar{c}d$ feasible
supercube $(a\bar{b}\bar{c}d,\ \bar{a}b\bar{c}\bar{d}) = \bar{c}$ infeasible
supercube $(a\bar{b}\bar{c}d,\ \bar{a}b\bar{c}d) = \bar{c}d$ feasible
supercube $(a\bar{b}\bar{c}d,\ ab\bar{c}d) = a\bar{c}d$ feasible
supercube $(a\bar{b}\bar{c}d,\ a\bar{b}\bar{c}\bar{d}) = a\bar{b}\bar{c}$ feasible

There are two choices that are favorable.
Let us assume that $a\bar{b}\bar{c}d$ is expanded with $\bar{a}b\bar{c}d$ resulting in the cube $\bar{c}d$.
Now, the free set is $\{7\}$. However, this column can't be raised.
The covered minterms $\bar{a}\bar{b}\bar{c}d$, $\bar{a}b\bar{c}d$, $ab\bar{c}d$ and $a\bar{b}\bar{c}d$ are removed.
Next, we have a choice either to expand $a\bar{b}c\bar{d}$ or $abcd$.
Let us expand $a\bar{b}c\bar{d}$. Free-set $= \{1, 4, 5, 8\}$
Intersecting with the offset, we can see that column 8 can't be raised. Thus, the overexp. cube $= \bar{d}$.
Thus, we need to check the minterms $\bar{a}\bar{b}\bar{c}\bar{d}$, $\bar{a}\bar{b}c\bar{d}$, $\bar{a}b\bar{c}\bar{d}$, $\bar{a}bc\bar{d}$, $abc\bar{d}$, $a\bar{b}\bar{c}\bar{d}$ for being feasibly covered.

supercube $(a\bar{b}c\bar{d},\ \bar{a}\bar{b}\bar{c}\bar{d}) = \bar{b}\bar{d}$ feasible
supercube $(a\bar{b}c\bar{d},\ \bar{a}\bar{b}c\bar{d}) = \bar{b}c\bar{d}$ feasible
supercube $(a\bar{b}c\bar{d},\ \bar{a}b\bar{c}\bar{d}) = \bar{d}$ infeasible
supercube $(a\bar{b}c\bar{d},\ \bar{a}bc\bar{d}) = c\bar{d}$ feasible
supercube $(a\bar{b}c\bar{d},\ abc\bar{d}) = ac\bar{d}$ feasible
supercube $(a\bar{b}c\bar{d},\ a\bar{b}\bar{c}\bar{d}) = a\bar{b}\bar{d}$ feasible

We have two choices $\bar{b}\bar{d}$ or $c\bar{d}$.
We will select to expand with $\bar{a}\bar{b}\bar{c}\bar{d}$ to get $\bar{b}\bar{d}$.

we remove the covered minterms $\bar{a}\bar{b}\bar{c}\bar{d}$, $\bar{a}\bar{b}cd$, $a\bar{b}\bar{c}\bar{d}$, $a\bar{b}c\bar{d}$.

Next, we expand the minterm $abcd$.

Free set = {1, 3, 5, 7}.

Intersecting with the offset, one can see that column 3 cannot be raised.

The overexpanded cube is b.

we need to check the minterms $\bar{a}b\bar{c}\bar{d}$, $\bar{a}bcd$, $\bar{a}bc\bar{d}$, $abc\bar{d}$ for being feasibly covered.

Supercube $(abcd, \bar{a}b\bar{c}\bar{d}) = b$ infeasible

Supercube $(abcd, \bar{a}bcd) = bcd$ feasible

Supercube $(abcd, \bar{a}bc\bar{d}) = bc$ feasible

supercube $(abcd, abc\bar{d}) = abc$ feasible

So, the cube bc is selected and the minterms $\bar{a}bcd$, $\bar{a}bc\bar{d}$, $abcd$, $abc\bar{d}$ are removed.

Now there is only one remaining minterm $\bar{a}b\bar{c}\bar{d}$ to be expanded.

Free set = {2, 3, 6, 8}.

Intersecting with the offset, we can see that column 2 cannot be raised.

overexpanded cube is $\bar{a}$.

There are no remaining cubes to be feasibly covered.

Raising column 3 produces $\bar{a}\bar{c}\bar{d}$ intersects with one cube.

Raising column 6 produces $\bar{a}b\bar{d}$ intersects with one cube.

Raising column 8 produces $\bar{a}b\bar{c}$ intersects with one cub.

Thus, any expand direction can be used.

Let us assume that we raise column 3 and we get the expanded cube $\bar{a}\bar{c}\bar{d}$.
The free set now is $\{6, 8\}$
Raising column 6 produces $\bar{a}\bar{d}$ intersects two cubes.
Raising column 8 produces $\bar{a}\bar{c}$ intersects two cubes.
So, we can expand in any direction.
we assume that we raise column 6 and we get the expanded cube $\bar{a}\bar{d}$,
The free set becomes empty as column 8 cannot be raised now.

Thus, the obtained expanded cover is
$\{ \bar{c}d, \bar{b}\bar{d}, bc, \bar{a}\bar{d} \}$

Comparing the obtained solution with Espresso expanded cover tool, we can see that the expanded cover by the tool is $\{ \bar{b}\bar{d}, \bar{c}d, bc, \bar{a}\bar{d} \}$ which is the same expanded cover we obtained. However, the order of cube expansion by the tool was $a\bar{b}c\bar{d}$, then $a\bar{b}\bar{c}d$, $abcd$, $\bar{a}b\bar{c}\bar{d}$. This is because there several implicants of the same weight.

(iv) Irredundant Procedure

The expanded cover is $\{\bar{b}\bar{d}, bc, \bar{c}d, \bar{a}\bar{d}\}$

First, we need to check whether each of these cubes is relatively essential.

- check if $\{bc, \bar{c}d, \bar{a}\bar{d}\}$ covers $\bar{b}\bar{d}$

$\{bc, \bar{c}d, \bar{a}\bar{d}\}_{\bar{b}\bar{d}} = \{c, 0, \bar{a}\}$ not tautology.

Thus, $\bar{b}\bar{d}$ is relatively essential.

- check if $\{\bar{b}\bar{d}, \bar{c}d, \bar{a}\bar{d}\}$ covers bc

$\{\bar{b}\bar{d}, \bar{c}d, \bar{a}\bar{d}\}_{bc} = \{0, 0, \bar{a}\bar{d}\}$ not tautology

Thus, bc is relatively essential.

- check if $\{\bar{b}\bar{d}, bc, \bar{a}\bar{d}\}$ covers $\bar{c}d$

$\{\bar{b}\bar{d}, bc, \bar{a}\bar{d}\}_{\bar{c}d} = \{0, 0, 0\}$ not tautology

Thus, $\bar{c}d$ is relatively essential.

- check if $\{\bar{b}\bar{d}, bc, \bar{c}d\}$ covers $\bar{a}\bar{d}$

$\{\bar{b}\bar{d}, bc, \bar{c}d\}_{\bar{a}\bar{d}} = \{\bar{b}, bc, 0\}$ not tautology

Thus, $\bar{a}\bar{d}$ is relatively essential.

Since all implicants are relatively essential, then the cover is irredundant.

This is consistent with what is produced by Espresso tool.

(v) Essential Prime Implicants

Prime implicants $= \{\bar{b}\bar{d}, bc, \bar{c}d, \bar{a}\bar{d}\}$

- checking $\bar{b}\bar{d}$

  $G = \{bc, \bar{c}d, \bar{a}\bar{d}\}$

  $G \# \bar{b}\bar{d} = \{bc, \bar{c}d, \bar{a}b\bar{d}\}$

  $H =$ Consensus $(\{bc, \bar{c}d, \bar{a}b\bar{d}\}, \bar{b}\bar{d})$

  $= \{c\bar{d}, \bar{b}\bar{c}, \bar{a}\bar{d}\}$

  Then, we check $\{c\bar{d}, \bar{b}\bar{c}, \bar{a}\bar{d}\}_{\bar{b}\bar{d}} = \{c, \bar{c}, \bar{a}\}$

  since it is tautology, this means that $\bar{b}\bar{d}$ is not essential.

- checking bc

  $G = \{\bar{b}\bar{d}, \bar{c}d, \bar{a}\bar{d}\}$

  $G \# bc = \{\bar{b}\bar{d}, \bar{c}d, \bar{a}\bar{b}\bar{d}, \bar{a}\bar{c}\bar{d}\}$

  $H =$ Consensus $(\{\bar{b}\bar{d}, \bar{c}d, \bar{a}\bar{b}\bar{d}, \bar{a}\bar{c}\bar{d}\}, bc)$

  $= \{c\bar{d}, bd, \bar{a}c\bar{d}, \bar{a}b\bar{d}\}$

  $\{c\bar{d}, bd, \bar{a}c\bar{d}, \bar{a}b\bar{d}\}_{bc} = \{\bar{d}, d, \bar{a}\bar{d}, \bar{a}\bar{d}\}$ this means that

  since it is tautology, bc is not essential.

- checking $\bar{c}d$

  $G = \{\bar{b}\bar{d}, bc, \bar{a}\bar{d}\}$

  $G \# \bar{c}d = \{\bar{b}\bar{d}, bc, \bar{a}\bar{d}\}$

Cons. $(\{\bar{b}\bar{d}, bc, \bar{a}\bar{d}\}, \bar{c}d) = \{\bar{b}\bar{c}, bd, \bar{a}\bar{c}\}$

  $\{\bar{b}\bar{c}, bd, \bar{a}\bar{c}\}_{\bar{c}d} = \{\bar{b}, b, \bar{a}\}$

  since it is tautology, this means that $\bar{c}d$ is not essential.

- checking $\bar{a}\bar{d}$

$G = \{\bar{b}\bar{d}, bc, \bar{c}d\}$

$G \# \bar{a}\bar{d} = \{a\bar{b}\bar{d}, bcd, abc, \bar{c}d\}$

Consensus $(\{a\bar{b}\bar{d}, bcd, abc, \bar{c}d\}, \bar{a}\bar{d})$

$= \{\bar{b}\bar{d}, \bar{a}bc, bc\bar{d}, \bar{a}\bar{c}\}$

Then, $\{\bar{b}\bar{d}, \bar{a}bc, bc\bar{d}, \bar{a}\bar{c}\}_{\bar{a}\bar{d}} = \{\bar{b}, bc, bc, \bar{c}\}$

since it is tautology, we conclude that $\bar{a}\bar{d}$ is not essential.

Thus, none of the implicants is essential.

(vi) Reduce procedure

Irredundant cover $= \{\bar{b}\bar{d}, bc, \bar{c}d, \bar{a}\bar{d}\}$

First, we compute the weight of each implicant.

	a	b	c	d	weight
$\bar{b}\bar{d}$	11	10	11	10	19
bc	11	01	01	11	18
$\bar{c}d$	11	11	10	01	18
$\bar{a}\bar{d}$	10	11	11	10	19
	43	33	33	32	

We will start by reducing the implicant with the largest weight. Thus, we can start by reducing either $\bar{b}\bar{d}$ or $\bar{a}\bar{d}$.

- We will attempt to reduce $\bar{a}\bar{d}$ first.

$\alpha = \bar{a}\bar{d}$

$Q = \{\bar{b}\bar{d}, bc, \bar{c}d\} = \bar{b}\bar{d} + bc + \bar{c}d$

$\bar{Q} = \bar{b}cd + b\bar{c}\bar{d}$

$\bar{Q}_\alpha = b\bar{c}$

$\tilde{\alpha} = \alpha \cap \text{supercube}(\bar{Q}_\alpha)$

$= \bar{a}\bar{d} \cap b\bar{c} = \bar{a}b\bar{c}\bar{d}$

Thus, the cube $\bar{a}\bar{d}$ is reduced to $\bar{a}b\bar{c}\bar{d}$

- <u>Reduce $\bar{b}\bar{d}$</u>

$\alpha = \bar{b}\bar{d}$

$Q = \{bc, \bar{c}d, \bar{a}b\bar{c}\bar{d}\} = bc + \bar{c}d + \bar{a}b\bar{c}d$

$= \bar{c}[d + \bar{a}bd] + c[b] = \bar{c}[d] + c[b]$

$\bar{Q} = \bar{c}\bar{d} + c\bar{b}$

$\bar{Q}_\alpha = \bar{c} + c = 1$

$\tilde{\alpha} = \alpha \cap \text{supercube}(\bar{Q}_\alpha) = \bar{b}\bar{d} \cap 1 = \bar{b}\bar{d}$

Thus, the cube $\bar{b}\bar{d}$ cannot be reduced.

- <u>Reduce bc</u>

$\alpha = bc$

$Q = \{\bar{b}\bar{d}, \bar{c}d, \bar{a}b\bar{c}\bar{d}\} = \bar{b}\bar{d} + \bar{c}d + \bar{a}b\bar{c}\bar{d}$

$= \bar{d}[\bar{b} + \bar{a}b\bar{c}] + d[\bar{c}]$

$\bar{Q} = \bar{d}[ba + bc] + dc$

$= ab\bar{d} + bc\bar{d} + cd$

$\bar{Q}_\alpha = a\bar{d} + \bar{d} + d = 1$

$\tilde{\alpha} = \alpha \cap \text{supercube}(\bar{Q}_\alpha) = bc \cap 1 = bc$

Thus, it cannot be reduced.

- <u>Reduce $\bar{c}d$</u>

$Q = \{\bar{b}\bar{d}, bc, \bar{a}b\bar{c}\bar{d}\} = \bar{b}\bar{d} + bc + \bar{a}b\bar{c}\bar{d}$

$= \bar{b}[\bar{d}] + b[c + \bar{a}\bar{c}\bar{d}]$

$= \bar{b}[\bar{d}] + b[c + \bar{a}\bar{d}]$

$\bar{Q} = \bar{b}d + b\bar{c}a + b\bar{c}d$

$\bar{Q}_\alpha = \bar{b} + ab + b = 1$

$\tilde{\alpha} = \alpha \wedge \text{supercube}(\bar{Q}_\alpha) = \bar{c}d$

Thus, it cannot be reduced.

The reduced cover is $\{\bar{a}b\bar{c}\bar{d}, \bar{b}\bar{d}, bc, \bar{c}d\}$.

(vii) Expand Procedure on reduced cover:

We first compute the weights of the given implicants.

	a	b	c	d	weight
$\bar{a}b\bar{c}\bar{d}$	10	01	10	10	13
$\bar{b}\bar{d}$	11	10	11	10	17
bc	11	01	01	11	17
$\bar{c}d$	11	11	10	01	17
	43	23	32	32	

Thus, the cube $\bar{a}b\bar{c}\bar{d}$ will be expanded first.

Free-set $= \{2, 3, 6, 8\}$. We can see that intersecting with the offset, column 2 cannot be raised. overexpanded cube is $\bar{a}$.

since none of the other cubes is covered by the over expanded cube, none of them is feasibly covered.

Raising column 3 produces $\bar{a}\bar{c}\bar{d}$ intersects one cube.

Raising column 6 produces $\bar{a}b\bar{d}$ intersects one cube.

Raising column 8 produces $\bar{a}b\bar{c}$ intersects one cube.

Let us assume that we raise column 3 and we get $\bar{a}\bar{c}\bar{d}$.

The free-set now is $\{6, 8\}$

Raising column 6 produces $\bar{a}\bar{d}$ intersects two columns.

Raising column 8 produces $\bar{a}\bar{c}$ intersects two columns.

Thus, any one can be selected.

If we raise column 6, we end up with the same cover we had before and the tool will terminate.

This is what is generated by the Espresso tool.

(viii) **VHDL Model**

```
library ieee;
use ieee.std_logic_1164.all;


entity hw2 is
port (a,b, c, d:in std_logic; f:out std_logic);
end hw2;
architecture  behav of hw2 is
begin
f <=     (not(a) and not(b) and not(c) and not(d))OR
    (not(a) and not(b) and not(c) and  d)OR
    (not(a) and not(b) and  c and not(d))OR
    (not(a) and  b and not(c) and not(d))OR
    (not(a) and  b and not(c) and  d)OR
    (not(a) and  b and  c and not(d))OR
    (not(a) and  b and  c and  d)OR
    ( a and not(b) and not(c) and not(d))OR
    ( a and not(b) and not(c) and  d)OR
    ( a and not(b) and  c and not(d))OR
    ( a and  b and not(c) and  d)OR
    ( a and  b and  c and not(d))OR
    ( a and  b and  c and  d);
end behav ;
```

Using Design Compiler and library and_or.lib with default constraints, we get the following circuit: