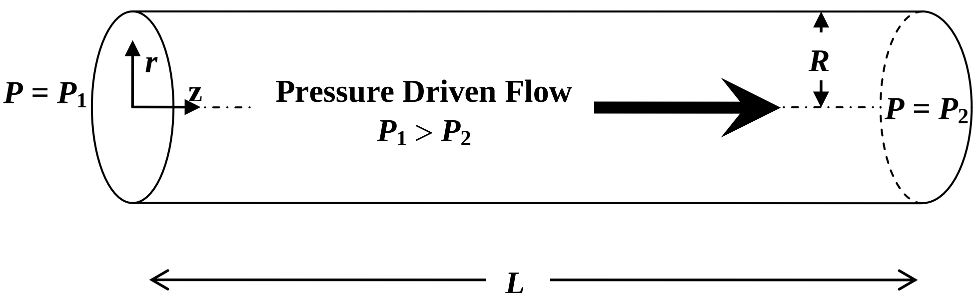
Pressure Driven Flow through a Horizantal Circular Pipe

A fluid of constant density ρ and viscosity μ flows through a horizontal pipe of radius R and length L shown in figure below. The pressures at the centers of the inlet and exit are p_1 and p_2, respectively. You may assume that the only nonzero velocity component is v_z, and that this not a function of the angular coordinate, θ.



Starting any further necessary assumptions, derive expressions for the following, in terms of any or all of R, L, p_1, p_2,ρ,μ, and the coordinates r, z, and θ:

a) Velocity Profile
b) Volumetric Flow Rate
c) Maximum Velocity
d) Mean Velocity
e) Shear Stress

<u>Solution:</u>

First we start with the continuity equation in cylindrical coordinates for incompressible fluid (the density is constant):

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial(rv_\theta)}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Since the flow is in the z-direction only, then we have only one component of the velocity $v_z \neq 0$ and $v_r = v_\theta = 0$, continuity equation simplifies to:

$$\frac{\partial v_z}{\partial z}=0$$

Conclusion the simplified continuity equation implies that v_z is not a function of z

$$v_z \neq f(z)$$

Also, for axi-symmetric problem, $v_z \neq f(\theta)$.

Second we use the Navier-Stokes equations in Cartesian coordinates:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial x}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial y}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r,$$
$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial x}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta,$$
$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial x}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z.$$

To simplify Navier-Stokes equations we can utilize the following results:

1. Steady state: $\frac{\partial(\text{any thing})}{\partial t}=0$
2. Axi-symmetric problem: $\frac{\partial(\text{any thing})}{\partial\theta}=0$
3. We have one component of the velocity $v_z \neq 0$ and $v_r = v_\theta = 0$
4. $v_z \neq f(z, \theta)$ it is only a function of y: $v_z = f(r)$.
5. $g_z = 0$

Therefore the N-S equations simplify to:

$$0=-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

In pressure driven flows like this problem the pressure changes linearly along the direction of the flow:

$$\frac{\partial p}{\partial z}=\frac{p_2-p_1}{L}=\frac{\Delta p}{L}$$

Velocity Profile:

Integrate simplified NS equation once:

$$\int d\left(r\frac{dv_z}{dr}\right) = \int \frac{1}{\mu}\frac{\Delta p}{L} r\,dr$$

$$r\frac{dv_z}{dr} = \frac{1}{\mu}\frac{\Delta p}{L}\frac{r^2}{2} + C_1$$

Integrate second time:

$$\int dv_z = \int \left(\frac{1}{\mu}\frac{\Delta p}{L}\frac{r}{2} + \frac{C_1}{r}\right) dr$$

$$\Rightarrow v_z = \frac{1}{\mu}\frac{\Delta p}{L}\frac{r^2}{4} + C_1 \ln(r) + C_2$$

To find the constants of integration apply the following Boundary Conditions:

$$r = 0 \qquad \frac{dv_z}{dr} = 0 \qquad (\text{Velocity is maximum at center of pipe})$$

$$r = R \qquad v_z = 0 \qquad (\text{No - Slip Boundary Condition})$$

$$\frac{dv_z}{dr} = \frac{1}{\mu}\frac{\Delta p}{L}\frac{r}{2} + \frac{C_1}{r}$$

$$0 = \frac{1}{\mu}\frac{\Delta p}{L}\frac{0}{2} + \frac{C_1}{0} \qquad \Rightarrow \qquad C_1 = 0 \ (\text{otherwise we have } \infty!)$$

$$\Rightarrow v_z = \frac{1}{\mu}\frac{\Delta p}{L}\frac{R^2}{4} + 0\ln(R) + C_2 \qquad \Rightarrow \quad C_2 = -\frac{1}{\mu}\frac{\Delta p}{L}\frac{R^2}{4}$$

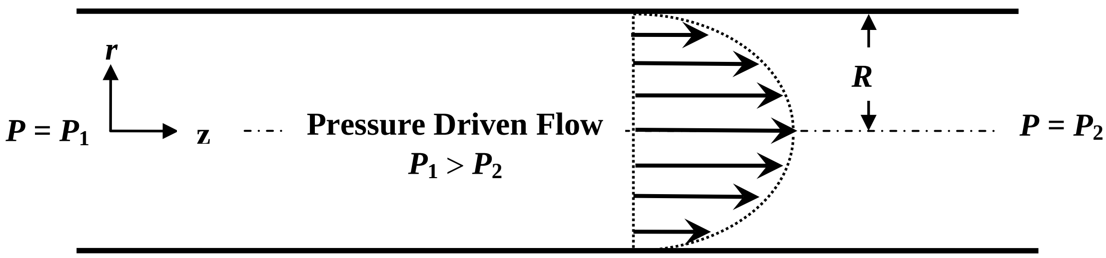
$$\Rightarrow v_z = \frac{1}{\mu}\frac{\Delta p}{L}\frac{r^2}{4} - \frac{1}{\mu}\frac{\Delta p}{L}\frac{R^2}{4}$$

Rearrange:

$$\Rightarrow v_z = \frac{1}{4\mu}\frac{\Delta p}{L}\left(r^2 - R^2\right)$$

The above equation is similar to an equation of a parabola and hence the velocity profile is called a parabolic velocity profile, see figure below:

Parabolic Velocity Profile



Volumetric Flow Rate:

$$Q = \int_A v_z \, dA$$

In cylindrical coordinates:

$$dA = r dr \, d\theta$$

$$Q = \int_0^{2\pi}\int_0^{R} v_z \, r dr \, d\theta = \int_0^{R} v_z \, 2\pi \, r dr$$

$$\begin{aligned} Q &= \int_0^{R} \frac{1}{4\mu}\frac{\Delta p}{L}\left(r^2 - R^2\right) 2\pi \, r dr \\ &= \frac{\pi}{2\mu}\frac{\Delta p}{L}\int_0^{R}\left(r^3 - R^2 r\right) dr \\ &= \frac{\pi}{2\mu}\frac{\Delta p}{L}\left(\frac{r^4}{4} - R^2\frac{r^2}{2}\right)_0^{R} \\ &= \frac{\pi}{2\mu}\frac{\Delta p}{L}\frac{-R^4}{4} \end{aligned}$$

$$Q = \frac{\pi R^4}{8\mu}\frac{-\Delta p}{L} \qquad \text{(Hagen Poiseuille Law)}$$

Maximum Velocity:

$$v_{Max} = v_z\big|_{r=0} = \frac{R^2}{4\mu}\frac{-\Delta p}{L}$$

Mean Velocity:

$$v_m = \frac{Q}{A} = \frac{\dfrac{-\pi R^4 \Delta p}{8\mu L}}{\pi R^2} = \frac{R^2}{8\mu}\frac{-\Delta p}{L} = \frac{v_{Max}}{2}$$

Shear Stress:

$$\begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} \left(2\dfrac{\partial v_r}{\partial r}\right) & \left(r\dfrac{\partial}{\partial r}\left(\dfrac{v_\theta}{r}\right) + \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta}\right) & \left(\dfrac{\partial v_z}{\partial r} + \dfrac{\partial v_r}{\partial z}\right) \\ \left(r\dfrac{\partial}{\partial r}\left(\dfrac{v_\theta}{r}\right) + \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta}\right) & \left(2\dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + 2\dfrac{v_r}{r}\right) & \left(\dfrac{\partial v_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta}\right) \\ \left(\dfrac{\partial v_z}{\partial r} + \dfrac{\partial v_r}{\partial z}\right) & \left(\dfrac{\partial v_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta}\right) & \left(2\dfrac{\partial v_z}{\partial z}\right) \end{bmatrix}$$

Recall simplifications:

1. Axi-symmetric problem: $\dfrac{\partial(\text{any thing})}{\partial\theta} = 0$
2. We have one component of the velocity $v_z \neq 0$ and $v_r = v_\theta = 0$
3. $v_z \neq f(z, \theta)$ it is only a function of y: $v_z = f(r)$.

This leads to the following simplifications:

$$\begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} 0 & 0 & \left(\dfrac{\partial v_z}{\partial r}\right) \\ 0 & 0 & 0 \\ \left(\dfrac{\partial v_z}{\partial r}\right) & 0 & 0 \end{bmatrix}$$

Therefore, the only nonzero stresses are: $\tau_{rz} = \tau_{zr} = \mu\left(\dfrac{\partial v_z}{\partial r}\right)$.

Recall, $v_z = \dfrac{1}{4\mu}\dfrac{\Delta p}{L}\left(r^2 - R^2\right)$

$$\Rightarrow \tau_{rz} = \frac{1}{2}\frac{\Delta p}{L} r$$

$$\Delta p = p_2 - p_1 = -ve \quad \Rightarrow \quad \tau_{rz} \quad \text{is} \quad -ve$$

Shear Stress Profile