Problem #1:

Given:
The figure shown
\[ r = 30 \text{ mm} \quad ; \quad G = 100 \text{ GPa} \]

Required:
Value and location of \( \tau_{\text{max}} \).
\( \varphi_F \)

Solution:

The system is statically indeterminate as it is fixed at A and B; thus, there are two unknowns \( T_A \) and \( T_B \) (reactions) and only one equilibrium equation (\( \Sigma T = 0 \)), as shown below in the FBD.

1. **Equilibrium**
\[
\Sigma T_{axis} = 0
\]

Since the two shafts are not on the same line, when we take \( \Sigma T \) about the axis of one of them, the reaction on the bearing of the other one will produce \( "T" \), and thus the reaction appears on the equation. Therefore, it is better to separate the two shafts from the gear and take each one separately as shown below.

\[
\Sigma T_{AE_{axis}} = 0 \Rightarrow T_A - 300 + F(0.1) = 0 \quad (1)
\]

\[
\Sigma T_{FB_{axis}} = 0 \Rightarrow T_B + F(0.07) = 0 \quad (2)
\]
From (2), \[ F = - \frac{T_B}{0.07} \Rightarrow \]
\[ T_A - 300 + \left( - \frac{T_B}{0.07} \right)(0.1) = 0 \Rightarrow \]
\[ T_A - \frac{10}{7}T_B - 300 = 0 \quad (3) \]

2) Geometric Compatibility:

The geometric compatibility of the gears will be used (as discussed in the previous HW).

\[ r_E \theta_E = r_F \theta_F \]
\[ \theta = \varphi \text{ in our case} \Rightarrow \]
\[ 0.1 \varphi_E = 0.07 \varphi_F \Rightarrow \varphi_E = 0.7 \varphi_F \quad (4) \]

From the boundary conditions,

\[ \varphi_E = \varphi_{E/A} = \varphi_{AE} \text{ as } A \text{ is fixed} \]
\[ \varphi_F = \varphi_{F/B} = \varphi_{FB} \text{ as } B \text{ is fixed} \]

Then, eq. (4) becomes \( \frac{TL}{J_G} \) \( \frac{AE}{AE} = 0.7 \left( \frac{TL}{J_G} \right) \) \( FB \) \( \Rightarrow \]

From the FBD shown

\[ T_{AE} = -T_A \]
\[ T_{FB} = T_B \]

Note that both \textit{internal} torques are assumed positive. (Why & How?!) \( J \) \& \( G \) are common in the two shafts, thus eq. (5) becomes \[ -1.5T_A = 0.7(0.8T_B) \Rightarrow \]
\[ T_B = -2.67857T_B \quad (6) \]

From eq. (6) into eq. (3), \[ T_A - \frac{10}{7}(-2.67857T_A) - 300 = 0 \Rightarrow \]
\[ T_A = 62.156 \text{ N.m} \Rightarrow \]
\[ T_B = -166.49 \text{ N.m} = 166.49 \text{ N.m} \]

Since \( r \) is the same for the two shafts, \( \tau_{\text{max}} \) \textit{will be} \( T_{\text{max}} = T_{FB} = T_B \Rightarrow \)
\[ \tau_{\text{max}} = \frac{T_{\text{max}}r_{\text{max}}}{J} = \frac{T_Br_{\text{out}}}{J} \Rightarrow \]
\[ \tau_{\text{max}} = \frac{166.49 \, (0.03)}{\frac{\pi}{2} (0.03)^4} \Rightarrow \]

\[ \tau_{\text{max}} = 3.926 \, \text{MPa @ outer radius in shaft FB} \]

((Do not worry about the sign!!))

\[ \varphi_F = \varphi_{F/B} = \varphi_{FB} \quad \text{as } F \text{ is fixed} \Rightarrow \]

\[ \varphi_F = \left(\frac{TL}{JG}\right)_{FB} = \left[ \frac{166.49 \, (0.8)}{\frac{\pi}{2} (0.03)^4 (100)(10)^9} \right] \Rightarrow \]

\[ \varphi_F = 1.0468 \times 10^{-3} \, \text{rad} = 0.05998^\circ \quad \text{CW} \]
Problem # 2:

Given:
The figure shown

Required:
\( T_A \); \( T_B \)
\( \varphi_{D/C} \); \( \varphi_{D/B} \)

Solution:

There are two reactions (at A and B) and only one static (equilibrium) equation (\( \Sigma T = 0 \)). Thus, the problem is statically indeterminate.

1. Equilibrium

From the FBD,
\[ \Sigma T_{\text{axis}} = 0 \Rightarrow \]
\[ T_A + T_B + 750 = 0 \quad \text{(1)} \]

2. Geometric Compatibility:

\[ \varphi_{A/B} \Rightarrow \sum \varphi_{A/B} = 0 \Rightarrow \]
\[ \varphi_{AC} + \varphi_{CD} + \varphi_{DB} = 0 \quad \text{(Why 3 segments?!) } \]
\[ \left( \frac{T_L}{I_G} \right)_{AC} + \left( \frac{T_L}{I_G} \right)_{CD} + \left( \frac{T_L}{I_G} \right)_{DB} = 0 \quad \text{(2)} \]

Since all segments are made of the same material (Steel with \( G = 75 \text{ GPa} \)), \( G \) can be dropped from the equation.

\( T \)'s can be obtained from the FBD's below.

\( T_{AC} = -T_A \)
\( T_{CD} = -T_A \)
\( T_{DB} = -T_A - 750 \)
Solution of HW # 8

Note that all internal torques are assumed (+). Also note that the right part of segment DB may be chosen as shown.

In this case $T_{DB} = + T_B$.

We can check this later.

Now, eq. (2) can be written as

\[
\frac{0.125 (-T_A)}{\pi \left( \frac{0.012}{2} \right)^4} + \frac{0.2 (-T_A)}{\pi \left( \frac{0.025}{2} \right)^4} + \frac{0.3 (-T_A - 750)}{\pi \left( \frac{0.025}{2} \right)^4} = 0 \Rightarrow
\]

\[
T_A = -78.816 \text{ N.m} = 78.816 \text{ N.m}
\]

(\text{as expected!})

\[
T_B = -671.18 = 671.18 \text{ N.m}
\]

(\text{as expected!})

Note that $T_B$ is noticeably bigger than $T_A$. Reasonable?! Why?!

\[
\varphi_{D/C} = \frac{TL}{JG}_{CD} = \frac{78.816 \times 0.2}{\pi \left( \frac{0.025}{2} \right)^4 \times (75)(10)^9} \Rightarrow
\]

\[
\varphi_{D/C} = 5.4805 \times 10^{-3} \text{ rad} = 0.3140^\circ
\]

\[
\varphi_{D/B} = \frac{TL}{JG}_{DB}
\]

\[
T_{DB} = -T_A - 750 = -(78.816) - 750 = -671.18 \text{ N.m} = T_B \quad \text{Ok!}
\]

\[
\varphi_{D/B} = \frac{671.18 \times 0.3}{\pi \left( \frac{0.025}{2} \right)^4 \times (75)(10)^9} \Rightarrow
\]

\[
\varphi_{D/B} = 0.070 \text{ rad} = 4.011^\circ
\]

Check:

When we add $\varphi_{C/A}$ to $\varphi_{D/C}$ we should end up with $\varphi_{D/A}$ which should equal to $\varphi_{D/B}$. ⇒

\[
\varphi_{C/A} = \frac{TL}{JG}_{AC} = \frac{-(-78.816)(0.125)}{\pi \left( \frac{0.012}{2} \right)^4 \times (75)(10)^9} = 0.064527 \text{ rad} = 3.6971^\circ
\]

\[
\varphi_{D/A} = \varphi_{C/A} + \varphi_{D/C} = 3.6971 + 0.3140 = 4.011^\circ = \varphi_{D/B} \quad \text{Ok!}
\]
Problem # 3:

Given:
The figure shown
Core: solid steel; \( d = 60 \text{ mm}; G = 80 \text{ GPa} \)
Tube: brass; \( G = 40 \text{ GPa} \)

Required:
Value and location of \( \tau_{\text{max}, \text{min}} \) in steel and brass
\( \tau \) distribution

Solution:

The problem is internally statically indeterminate; that is the 5 \( \text{kN} \cdot \text{m} \) – \( T \) is carried by steel and brass, but we do not know \( T_{\text{steel}} \) & \( T_{\text{brass}} \).

We know that the shear strain (\( \gamma \)) must be continuous through the radius as shown below.

For the shear stress (\( \tau \)), it is equal to \( G \gamma \). Since \( G \) changes on the border of steel/brass, \( \tau \) will be discontinuous at that point.

From the FBD (equilibrium)

\[
T_{\text{st}} + T_{\text{br}} + 5000 = 0 \quad (1)
\]

From the geometric compatibility,

\[
\varphi_{\text{st}} = \varphi_{\text{br}} \quad (2)
\]

((as steel and brass are bounded together, they must have the same rotation.))
From material behavior,

\[ \varphi = \frac{T_l}{JG} \quad (3) \]

From (3) into (2),

\[ \left( \frac{T_l}{JG} \right)_{st} = \left( \frac{T_l}{JG} \right)_{br} \Rightarrow \]

\[ \frac{T_{st}L}{\frac{\pi}{2}\left(\frac{60}{2}\right)^4(80)(10)^9} = \frac{T_{br}L}{\frac{\pi}{2}\left[\left(\frac{100}{2}\right)^4 - \left(\frac{60}{2}\right)^4\right](40)(10)^9} \Rightarrow \]

\[ T_{br} = 3.35802T_{st} \quad (4) \]

From (4) into (1), \( \Rightarrow T_{st} = 1147.31 \text{ N.m} \)

\( \Rightarrow \) into (4) \( \Rightarrow T_{br} = 3852.69 \text{ N.m} \)

Note that to take advantages of both materials, it is better to have the brass as the core (inner) material and the steel as the sleeve/tube (outer) material. (Why & How?!) \( \tau_{max} = \frac{T_{max}r_{max}}{J} \)

Since we have only one external \( T, \) \( T \) will be constant, and \( \tau_{max} \) @ \( r_{max} \) and \( \tau_{min} \) @ \( r_{min} \).

Thus,

\[ \tau_{max}^{st} = \frac{T_{st}r_{max}}{J_{st}} = \frac{1147.31(0.03)}{\frac{\pi}{2}(0.03)^4} \]

\[ \tau_{max}^{st} = 27.05 \text{ MPa} \]

\[ \tau_{min}^{st} = \frac{T_{st}r_{min}}{J_{st}} = \frac{T(0)}{J} \]

\[ \tau_{min}^{st} = 0 \]

\[ \tau_{max}^{br} = \frac{3852.69(0.05)}{\frac{\pi}{2}(0.05)^4 - (0.03)^4} \Rightarrow \]

\[ \tau_{max}^{br} = 22.54 \text{ MPa} \]
\[
\tau_{\text{br}}^{\text{min}} = \frac{3852.69 (0.03)}{\pi \left( (0.05)^4 - (0.03)^4 \right)} \Rightarrow
\]

\[\tau_{\text{br}}^{\text{min}} = 13.53 \text{ MPa}\]

**Check:**

\[\gamma_{\text{st}} = \gamma_{\text{br}} @ r = 0.03 \Rightarrow \left( \frac{\tau}{G} \right)_{\text{st}} = \left( \frac{\tau}{G} \right)_{\text{br}} \Rightarrow\]

\[\frac{\tau_{\text{st}}}{\tau_{\text{br}}} = \frac{G_{\text{st}}}{G_{\text{br}}} \Rightarrow\]

\[\frac{\tau_{\text{st}}}{\tau_{\text{br}}} = \frac{27.05}{13.53} = 2.00 \quad \text{OK}\]

\[\frac{G_{\text{st}}}{G_{\text{br}}} = \frac{80}{40} = 2.00\]

\(\tau - \text{distribution in MPa}\)
Problem # 4:

**Given:**

The figure shown

\[ \tau_{allow} = 120 \text{ MPa}; \quad \varphi^A_{max} = 1^\circ; \]
\[ \varphi^B_{max} = 0.5^\circ; \quad G = 100 \text{ GPa} \]

**Required:**

Maximum allowable \( T \)

**Solution:**

Here, we have four criteria (how?!) we need to satisfy. We calculate \( T_{max}^{allow} \) for each, and then we choose the smallest value for the answer \( T_{max}^{allow} \). (Why?! see previous HW!)

The problem is statically determinate so that we can find the internal \( T \) directly. We have two segments \( AB \) and \( BC \). (Why?!)

From the FBD’s,

\[ T_{AB} = T \]
\[ T_{BC} = 2T \]

For segment \( AB \), set \( \tau_{max} \equiv 120 \text{ MPa} \).

\[ \tau_{max} = \frac{T_{AB} \tau_{max}}{I_{AB}} = \frac{T_{max} (0.035)}{\frac{\alpha}{2} (0.035)^4} \equiv 120 (10)^6 \]

\[ \Rightarrow T_{max}^1 = 8.082 \text{ kN.m} \]

For segment \( BC \), set \( \tau_{max} \equiv 120 \text{ MPa} \).

\[ \tau_{max} = \frac{4.81 T_{BC}}{\alpha^3} = \frac{4.81 (2T_{max})}{(0.1)^3} \equiv 120 (10)^6 \]

\[ \Rightarrow T_{max}^2 = 12.47 \text{ kN.m} \]

Now consider \( \varphi^B \), then \( \varphi^A \). (Why?!)
\[ \varphi^B = \frac{7.10 \cdot T_{BC} \cdot L_{BC}}{a^4 G} = \frac{7.10 \cdot (2T_{max}) \cdot (0.8)}{(0.1)^4 \cdot (100)^9} \equiv 0.5 \left( \frac{\pi}{180} \right) \]
\[ \Rightarrow T_{max}^3 = 7.682 \text{ kN.m} \]

\[ \varphi^A = \sum \varphi = \left( \frac{T_L}{J_G} \right)_{AB} + \left( \frac{7.10 \cdot T_L}{aG} \right)_{BC} = \frac{T_{max} \cdot (0.6)}{2 \cdot (0.035)^4 \cdot (100)^9} + \frac{7.10 \cdot (2T_{max}) \cdot (0.8)}{(0.1)^4 \cdot (100)^9} \]
\[ 2.54542 \times 10^{-6} \; T_{max} + 1.136 \times 10^{-6} \; T_{max} \equiv 1 \left( \frac{\pi}{180} \right) \]
\[ \Rightarrow T_{max}^4 = 4.741 \text{ kN.m} \]

\[ T_{max}^4 \text{ due to } \varphi_{max}^A \text{ controls. (Why?!) } \]

\[ \Rightarrow \quad T_{allow} = 4.741 \text{ kN.m} \]
Problem #5:

Given:

The figure shown

\(a = 30 \text{ mm};\ b = 15 \text{ mm};\ T = 80 \text{ N.m}\)

Required:

\(\tau_{\text{max}}\) in the two sections

Efficiency of circular section compared with the elliptical one.

Solution:

\[
\tau_{\text{max}}^{\text{circ.}} = \frac{T \tau_{\text{max}}}{J} = \frac{80 (0.03)}{\pi (0.03)^4} \Rightarrow \tau_{\text{max}}^{\text{circ.}} = 1.886 \text{ MPa}
\]

\[
\tau_{\text{max}}^{\text{ell.}} = \frac{2T}{\pi ab^2} = \frac{2 (80)}{\pi (0.03)(0.015)^2} \Rightarrow \tau_{\text{max}}^{\text{ell.}} = 7.545 \text{ MPa}
\]

\[
\% \text{ max efficiency (circ./ell.)} = \frac{\tau_{\text{max}}^{\text{ell.}} - \tau_{\text{max}}^{\text{circ.}}}{\tau_{\text{max}}^{\text{circ.}}} \times 100
\]

\[
= \frac{7.545 - 1.886}{1.886} \times 100
\]

\[\text{Efficiency} = 300\ \%\]
Problem #6:

Given:
The cross-section of a shaft shown

\[ T = 300 \, \text{N.m}; \quad G = 100 \, \text{GPa} \]

Required:
Value and location of \( \tau_{\text{max}} \)

\[ \frac{d\varphi}{dz} \]

Solution:
The section is “thin-walled closed”. \( \Rightarrow \)

\[ \tau = \frac{T}{2tA_m} \]

\( A_m \) is the area contained within the mean perimeter (not material area) as shown in the below figure.

\[
A_m = A_1 + A_2 \\
= \frac{0.7 + 2 \times 0.7}{2} (1.396) + \frac{\pi}{2} (0.7)^2 \\
= 2.23549 \, \text{m}^2
\]

Since \( t \) is in the denominator of the \( \tau \) formula, \( \tau_{\text{max}} \) will be at \( t_{\text{min}} = 8 \, \text{mm} \) \( \Rightarrow \)

\[
\tau_{\text{max}} = \frac{300 \times (10)^3}{2 \times (0.008) \times (2.23549)} \Rightarrow \\
\tau_{\text{max}} = 8.387 \, \text{MPa} @ \text{the 8-mm thickness}
\]
\[
\frac{d\varphi}{dz} = \frac{T}{4A_m^2 G} \int \frac{ds}{t} \\
= \frac{T}{4A_m^2 G} \sum_{i=1}^{4} \frac{s_i}{t_i} \quad \text{in our case (Why?!)}
\]

\(S_1 = 0.7 \, m \quad ; \quad t_1 = 0.008 \, m\)

\(S_2 = \sqrt{(1.396)^2 + (0.35)^2} \]
\n\(= 1.43921 \, m = S_4\)

\(t_2 = t_4 = 0.01 \, m\)

\(S_3 = \pi r = \pi (0.7) = 2.19911 \, m \quad ; \quad t_3 = 0.008 \, m\)

\[
\frac{d\varphi}{dz} = \frac{300 (10)^3}{4 \cdot (2.23549)^2(100)(10)^9} \left( \frac{0.7}{0.008} + 2 \frac{1.43921}{0.01} + \frac{2.19911}{0.008} \right)
\]

\[
\frac{d\varphi}{dz} = 9.759 \, (10)^{-5} \, rad/m = 5.591 \, (10)^{-3} \, \circ/m
\]