

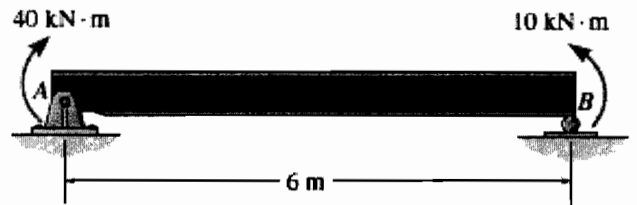
Problem # 1:

Given:

The beam shown

$$E = 200 \text{ GPa}$$

$$I = 39.9 \times 10^6 \text{ m}^4$$



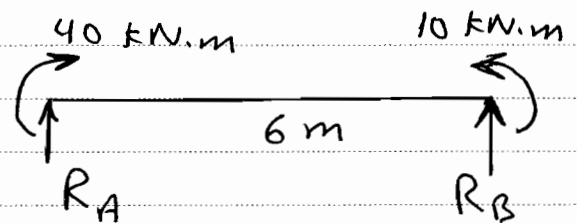
Required:

Maximum deflection

Solution:

We will start with the moment equation.
We may start with the load equations.

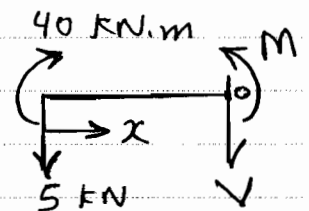
We need to find R_A . (Why?!))



In FBD ①,

$$\begin{aligned} \uparrow \sum M_B = 0 &\Rightarrow -40 + 10 - 6R_A = 0 && \text{FBD ①} \\ \Rightarrow R_A &= -5 \text{ kN} = 5 \text{ kN} \downarrow \end{aligned}$$

To get the moment equation, we draw FBD ② by taking a section "between A and B". Note that there is only one moment equation. (How?!))



$$\uparrow \sum M_o = 0 \Rightarrow M = 40 - 5x = EI \frac{d^2v}{dx^2} \quad \text{FBD ②} \quad (\text{kN}\cdot\text{m})$$

$$\text{Slope} = \theta = \frac{dv}{dx} = \int \frac{M}{EI} dx \Rightarrow$$

$$EI \frac{dv}{dx} = \int (40 - 5x) dx = 40x - \frac{5}{2}x^2 + C_1$$

$$EI v = \int EI \frac{dv}{dx} dx = \int EI (40x - \frac{5}{2}x^2 + C_1) dx \Rightarrow$$

Solution of HW # 15

$$EIv = 20x^2 - \frac{5}{6}x^3 + C_1x + C_2$$

Boundary Conditions (B.C.s):

We have 2 B.C.s. and 2 unknowns (C_1 and C_2). Thus, we can find C_1 & C_2 .

$$v(0) = 0 \quad \left[\text{That is the deflection is zero at } x=0 \text{ (A).} \right] \Rightarrow$$

$$0 = 0 + 0 + 0 + C_2 \Rightarrow C_2 = 0$$

$$v(6) = 0 \Rightarrow 0 = 20(6)^2 - \frac{5}{6}(6)^3 + C_1(6) \Rightarrow$$

$$C_1 = -90$$

"with appropriate units"!

Thus,

$$EIv = 20x^2 - \frac{5}{6}x^3 - 90x$$

To get v_{max} , we set $\frac{dv}{dx} = 0 \Rightarrow$ get x . (Why?)

$$\Rightarrow \frac{dv}{dx} EI = 40x - \frac{5}{2}x^2 - 90 = 0 \Rightarrow$$

$$x^2 - 16x + 36 = 0 \Rightarrow x = 13.292 \text{ or } 2.7085 \text{ m}$$

X (outside the range) ✓

$$\Rightarrow v_{max} = \left[20(2.7085)^2 - \frac{5}{6}(2.7085)^3 - 90(2.7085) \right] /$$

$$200(10)^9(39.9)(10)^{-6} \text{ (10)}^3$$

$$\Rightarrow v = -0.014236 \text{ m}$$

KN \rightarrow N ↓

$$\Rightarrow \boxed{v = 14.24 \text{ mm } \downarrow \text{ @ } x = 2.709 \text{ m}}$$

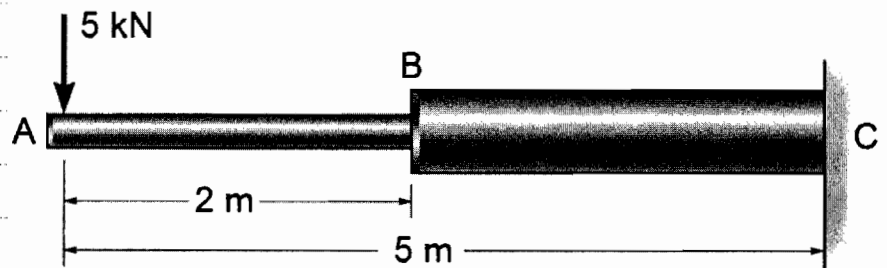
"Very small as in most applications"

Solution of HW # 15

Problem # 2:

Given:

The beam shown

 $E = \text{constant}$ $I = I_0$ for AB $= 2I_0$ for BC

Required:

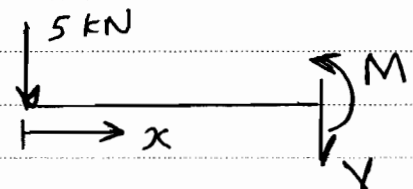
Deflection at B in terms of EI_0

Solution:

We will start with the moment equation. Note that there is only one moment equation for both AB and BC; however, there is a "discontinuity" in I at B. Thus, we need to divide the domain into two parts/segments (from A to B and from B to C) and use the "concept of continuity" in slope and deflection at B. \Rightarrow

From the FBD,

$$M = -5x \quad (\text{KN}\cdot\text{m})$$

AB ($0 \leq x \leq 2$):

FBD

$$EI_0 \frac{d^2v}{dx^2} = M = -5x$$

$$EI_0 \frac{dv}{dx} = \int M dx = -\frac{5}{2}x^2 + C_1 \quad (1)$$

$$EI_0 v = \int \left(-\frac{5}{2}x^2 + C_1\right) dx = -\frac{5}{6}x^3 + C_1 x + C_2 \quad (2)$$

BC ($2 \leq x \leq 5$):

$$2EI_0 \frac{dv}{dx} = M = -5x \quad \Rightarrow$$

$$2EI_0 \frac{dv}{dx} = \int (-5x) dx = -\frac{5}{2}x^2 + C_3 \quad (3)$$

$$2EI_0 v = \int \left(-\frac{5}{2}x^2 + C_3\right) dx = -\frac{5}{6}x^3 + C_3x + C_4 \quad (4)$$

B.C.s :

$$v(5) = v_c = 0 \quad \text{on range BC [eq. (4)]}$$

$$\frac{dv}{dx}(5) = \theta_c = 0 \quad \text{on range BC [eq. (3)]}$$

Continuity Conditions (C.C.s) :

$$v_{AB} = v_{BC} \quad @ \quad B$$

$$\left(\frac{dv}{dx}\right)_{AB} = \left(\frac{dv}{dx}\right)_{BC} \quad @ \quad B$$

Now, we have 4 B.C.s & C.C.s. and 4 unknown constants. We can find them.

$$v(5) = 0 \quad \text{in eq. (4)} \Rightarrow -\frac{5}{6}(5)^3 + 5C_3 + C_4 = 0 \quad (5)$$

$$\frac{dv}{dx}(5) = 0 \quad \text{in eq. (3)} \Rightarrow -\frac{5}{2}(5)^2 + C_3 = 0 \Rightarrow$$

$$C_3 = 62.5 \Rightarrow \text{into eq. (5)} \Rightarrow C_4 = -625/3$$

$$\left(\frac{dv}{dx}\right)(2)_{AB} = \left(\frac{dv}{dx}\right)(2)_{BC} \quad [\text{eq. (1) = eq. (3) @ B}]$$

$$\Rightarrow \frac{1}{EI_0} \left[-\frac{5}{2}(2)^2 + C_1\right] = \frac{1}{2EI_0} \left[-\frac{5}{2}(2)^2 + 62.5\right]$$

$$\Rightarrow C_1 = 36.25$$

$$v(2)_{AB} = v(2)_{BC} \quad [\text{eq. (2) = eq. (4) @ B}] \Rightarrow$$

Solution of HW # 15

$$\frac{1}{EI_0} \left[-\frac{5}{6} (2)^3 + 36.25 (2) + C_2 \right] = \frac{1}{2EI_0} \left[-\frac{5}{6} (2)^3 + 62.5 (2) - \frac{625}{3} \right]$$

$$\Rightarrow C_2 = -665/6$$

$$v_B = v(2) \text{ on AB} \quad \text{or} \quad \text{on BC}$$

Use AB. \Rightarrow Eq. (2) \Rightarrow

$$v_B = \left[-\frac{5}{6} (2)^3 + 36.25 (2) - 665/6 \right] / EI_0 \Rightarrow$$

$$v_B = -45 / EI_0$$

check using eq. (4) [BC] :

$$v_B = \left[-\frac{5}{6} (2)^3 + 62.5 (2) - 625/3 \right] / 2EI_0 \Rightarrow$$

$$v_B = -45 / EI_0 \quad \text{ok!}$$

Note that if we only need v_B , we can use eq. (4) after finding C_3 and C_4 from the B.C.s. Thus, no need for the C.C.s. to find C_1 and C_2 . In this case, we can not "check". However, if we need the eqs. of the elastic curves (def. eqs.), then we do as we did above.

Problem # 3:

Given:

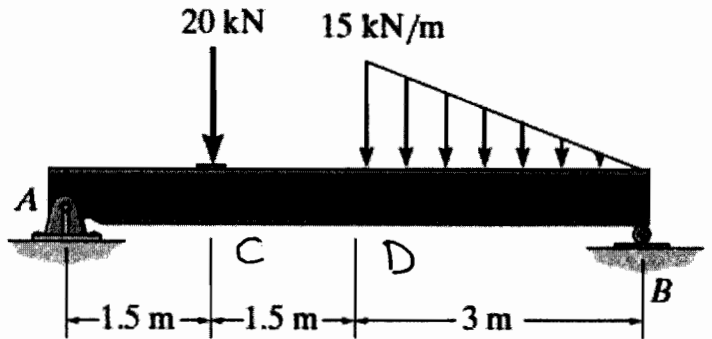
The beam shown

$$E = 200 \text{ GPa}$$

$$I = 65(10)^6 \text{ mm}^4$$

Required:

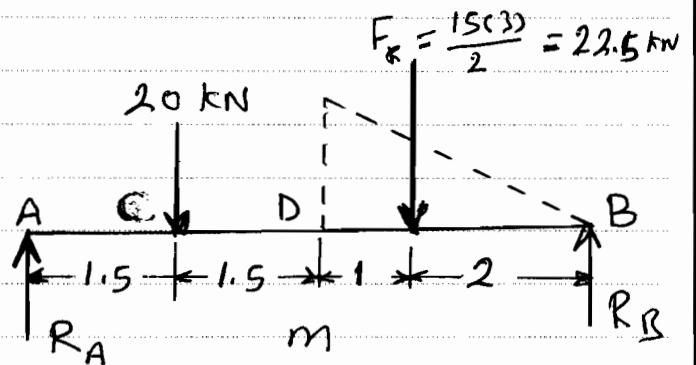
Maximum deflection



Solution:

Starting with the moment equation, we need R_A in

FBD ①. \Rightarrow



$$\uparrow \Sigma M_B = 0 \Rightarrow$$

$$-6R_A + 20(4.5) + 22.5(2) = 0 \Rightarrow$$

$$R_A = 22.5 \text{ kN}$$

FBD ①

To get the moment equation, we make a section in the last segment and draw FBD ② for the left part.

$$\frac{w(x)}{15} = \frac{6-x}{3} \Rightarrow$$

$$w(x) = 5(6-x)$$

$$F_1 = 15(x-3)$$

$$F_2 = (15-w) \left(\frac{x-3}{2} \right)$$

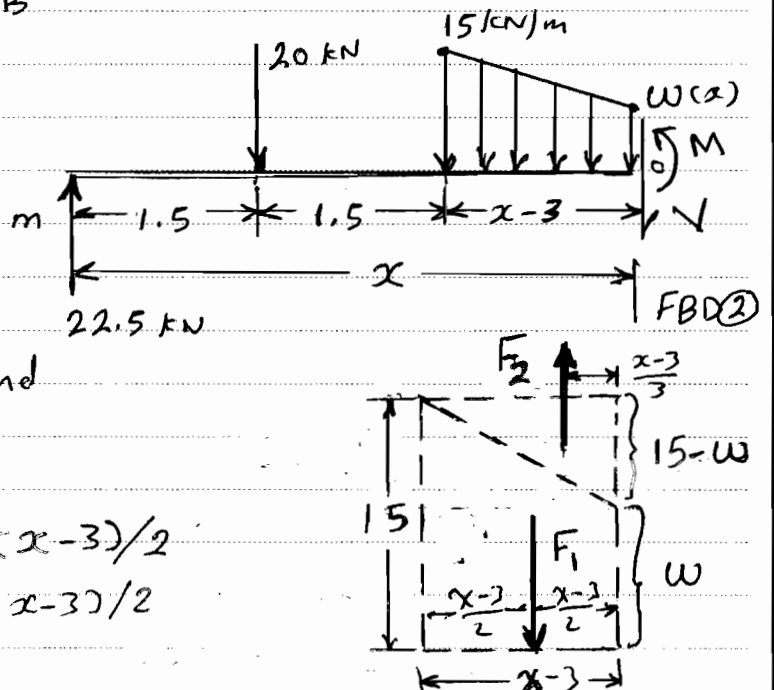
The locations of F_1 and F_2 are as shown.

$$F_2 = (15-w) \left(\frac{x-3}{2} \right)$$

$$= [15 - 5(6-x)] (x-3)/2$$

$$= 5[(3-6+x)] (x-3)/2$$

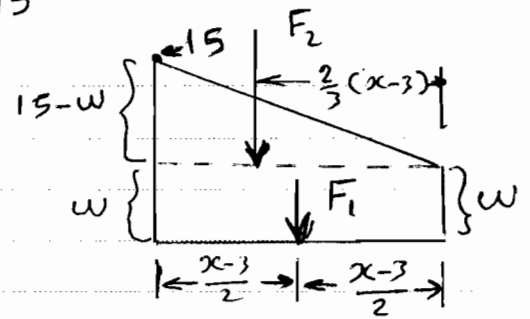
$$= \frac{5}{2} (x-3)^2$$



Solution of HW # 15

Try taking F_1 and F_2 as shown! \Rightarrow

What did you find?



$$\uparrow \Sigma M_o = 0 \Rightarrow$$

$$M - 22.5x + 20(x - 1.5)$$

$$+ 15(x-3)\left(\frac{x-3}{2}\right) - \frac{5}{2}(x-3)\left(\frac{x-3}{3}\right) = 0$$

$$F_1 = w(x-3) \downarrow$$

$$F_2 = (15-w)(x-3)/2 \downarrow$$

Moving the terms to the other side and using the singularity function form, we get:

$$M = 22.5 \langle x-0 \rangle^1 - 20 \langle x-1.5 \rangle^1 - \frac{15}{2} \langle x-3 \rangle^2 + \frac{5}{6} \langle x-3 \rangle^3$$

$$EI \frac{d^2 v}{dx^2} = M \Rightarrow EI \frac{dv}{dx} = \int M dx \Rightarrow$$

$$EI \frac{dv}{dx} = 11.25 \langle x-0 \rangle^2 - 10 \langle x-1.5 \rangle^2 - \frac{15}{6} \langle x-3 \rangle^3 + \frac{5}{24} \langle x-3 \rangle^4 + C_1$$

$$EI v = 3.75 \langle x-0 \rangle^3 - \frac{10}{3} \langle x-1.5 \rangle^3 - \frac{15}{24} \langle x-3 \rangle^4 + \frac{1}{24} \langle x-3 \rangle^5 + C_1 x + C_2$$

B.C.s.: $v(0) = 0 \Rightarrow 0 - 0 - 0 + 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$

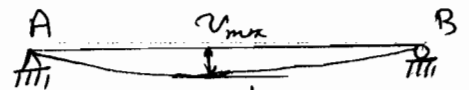
Note that the value within $\langle x-a \rangle$ can never be negative.

If so, it is ZERO! (Why and how?)

$$v(6) = v_B = 0 \Rightarrow 3.75(6)^3 - \frac{10}{3}(4.5)^3 - \frac{15}{24}(3)^4 + \frac{1}{24}(3)^5 + 6C_1 = 0$$

$$\Rightarrow C_1 = -77.625 \quad \text{C units in kN, m}^3$$

$$v_{max} \text{ is @ } \frac{dv}{dx} = 0$$



(OR at the free end if there is.) $\frac{dv}{dx} = 0$

$$\Rightarrow \text{Set } \frac{dv}{dx} = 0 \Rightarrow 11.25x^2 - 10 \langle x-1.5 \rangle^2 - \frac{15}{6} \langle x-3 \rangle^3 + \frac{5}{24} \langle x-3 \rangle^4 - 77.625 = 0$$

We can either find the four roots of this 4th degree equation, or assume the root within our range ($0 \leq x \leq 6$)

Solution of HW # 15

is between 0 and 3 m (as there is only one possible root within this range, as seen in the figure above). In this case, we have to check our assumption ($x \leq 3$). This will simplify our solution as the 3rd and 4th order terms are dropped. (How?!) \Rightarrow

$$11.25 x^2 - 10(x-1.5)^2 - 77.625 = 0 \Rightarrow$$

$$x = 2.96997 \quad \text{or} \quad -26.96997$$

« ok »

« not possible »

$\Rightarrow x_m = 2.970 \text{ m}$ « within our range 0-3 » \Rightarrow ok

Thus v_{max} @ $x_m = 2.970 \text{ m}$ \Rightarrow

$$v_{max} = \left[3.75 x_m^3 - \frac{10}{3} (x_m - 1.5)^3 - 77.625 x_m \right] (10)^3 / 200(10)^9 \cdot 65(10)^6$$

\Rightarrow

$$v_{max} = -0.010992 \text{ m} = 10.99 \text{ mm} \downarrow$$

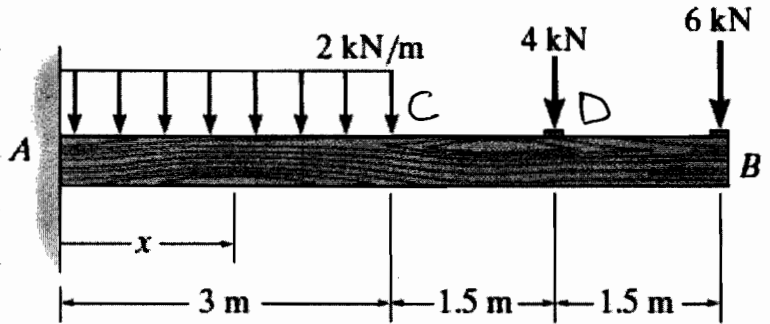
Again, very small compared with $L = 6 \text{ m}$

Problem # 4:

Given:

The beam shown

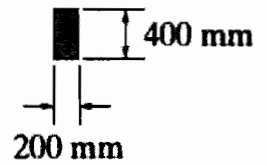
$E = 12 \text{ GPa}$



Required:

The eq. of the elastic curve (deflection)

Deflection & slope @ B



Solution:

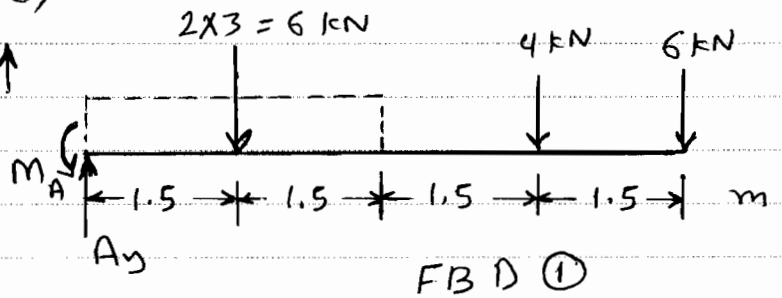
First, we need to find the reactions. From FBD ①,

$\uparrow \sum F_y = 0 \Rightarrow A_y = 16 \text{ kN} \uparrow$

$\curvearrowright \sum M_A = 0 \Rightarrow$

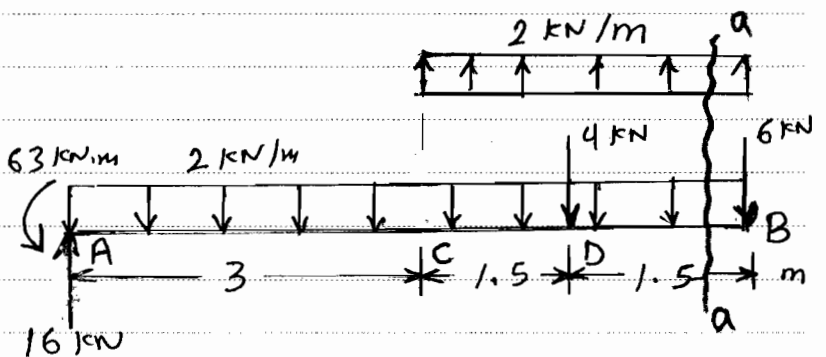
$M_A - 6(1.5) - 4(4.5) - 6(6) = 0$

$\Rightarrow M_A = 63 \text{ kN.m} \downarrow$



Using the singularity function, you need to remember that once the distributed load starts (from anywhere/x), it has to go all the way up to the end. (Why?) Thus, the 2 kN/m - load has to go from A to B. Therefore, we have to add an upward load equals to 2 kN/m from C to B, thus making the total loads equal to the original loading. This is shown in the figure below.

Now, we make a section (a-a) in the last segment (DB), and draw a FBD by taking



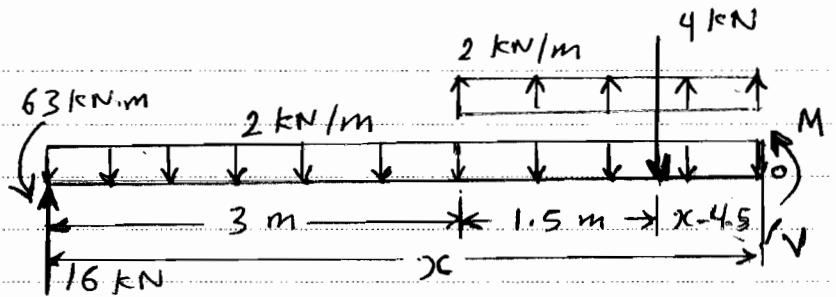
Solution of HW # 15

the left part, as shown FBD ②.

Note that any concentrated load/moment

(whether applied or reaction) does

not appear! Does it mean "it does not affect the beam"?! NO!! HOW??!!



FBD ②

3) $\Sigma M_o = 0 \Rightarrow$ In a singularity function form,

$$M = 16 \langle x-0 \rangle^1 - 63 \langle x-0 \rangle^0 - \frac{2}{2} \langle x-0 \rangle^2 + \frac{2}{2} \langle x-3 \rangle^2 - 4 \langle x-4.5 \rangle^1$$

$$EI \frac{dv}{dx} = 8 \langle x-0 \rangle^2 - 63 \langle x-0 \rangle^1 - \frac{1}{3} \langle x-0 \rangle^3 + \frac{1}{3} \langle x-3 \rangle^3 - 2 \langle x-4.5 \rangle^2 + C_1$$

$$EI v = \frac{8}{3} \langle x-0 \rangle^3 - \frac{63}{2} \langle x-0 \rangle^2 - \frac{1}{12} \langle x-0 \rangle^4 + \frac{1}{12} \langle x-3 \rangle^4 - \frac{2}{3} \langle x-4.5 \rangle^3 + C_1 x + C_2$$

B.c.s.: $\frac{dv}{dx}$ and $v = 0$ @ $x = 0$ (A) \Rightarrow

$$\frac{dv}{dx}(0) = 0 - 0 - 0 + 0 - 0 + C_1 = 0 \Rightarrow C_1 = 0$$

$$v(0) = 0 = 0 - 0 - 0 + 0 - 0 + 0 + C_2 \Rightarrow C_2 = 0$$

Thus,

$$v = \frac{1}{EI} \left[\frac{8}{3} \langle x-0 \rangle^3 - \frac{63}{2} \langle x-0 \rangle^2 - \frac{1}{12} \langle x-0 \rangle^4 + \frac{1}{12} \langle x-3 \rangle^4 - \frac{2}{3} \langle x-4.5 \rangle^3 \right]$$

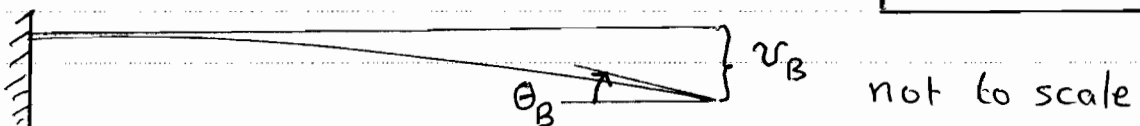
At B, $x = 6\text{ m}$; $E = 12(10)^9 \text{ N/m}^2$

$$I = \frac{bh^3}{12} \Rightarrow I = \frac{(0.2)(0.4)^3}{12} = \frac{3.2}{3}(10)^{-3} \text{ m}^4$$

$$\text{slope} = \theta_B = \frac{dv}{dx} = \left[8(6)^2 - 63(6) - \frac{1}{3}(6)^3 + \frac{1}{3}(3)^3 - 2(1.5)^2 \right] (10)^3 / EI$$

$$\Rightarrow \theta_B = -0.01230 \text{ rad} = -0.7050^\circ \quad \text{"cw"}$$

$$v_B = \left[\frac{8}{3}(6)^3 - \frac{63}{2}(6)^2 - \frac{1}{12}(6)^4 + \frac{1}{12}(3)^4 - \frac{2}{3}(1.5)^3 \right] / EI \Rightarrow v_B = -0.05168 \text{ m} \downarrow$$



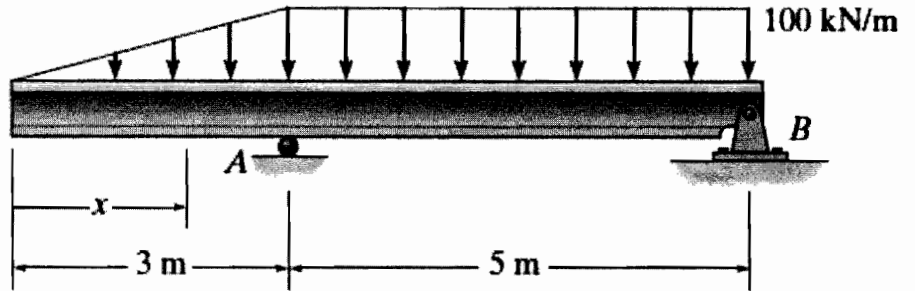
Solution of HW # 15

Problem # 5:

Given:

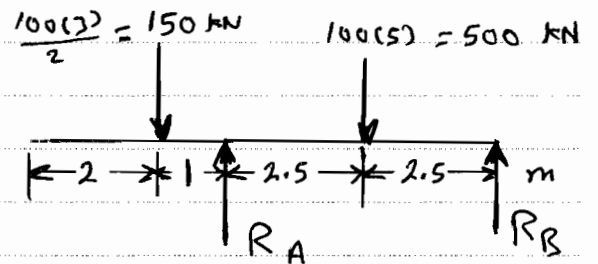
The beam shown

$EI = \text{Constant}$



Required:

Eq. of elastic curve (deflection)

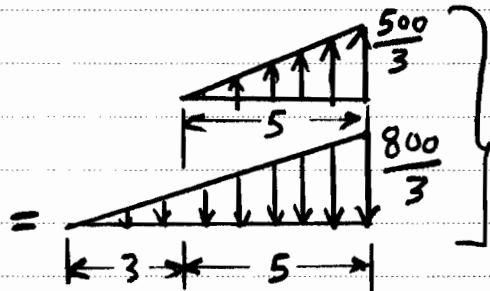
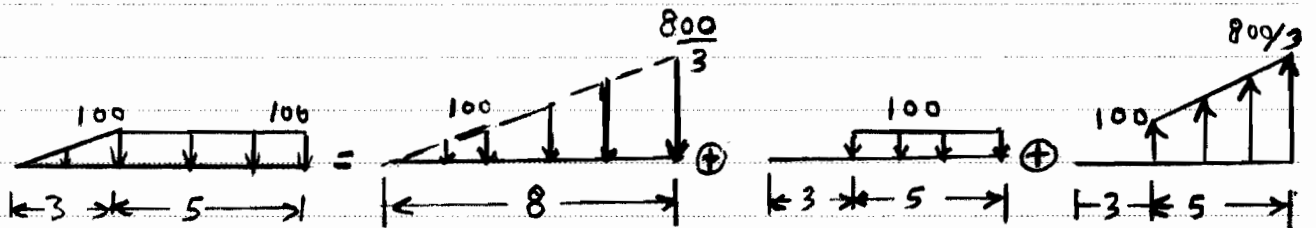


Solution:

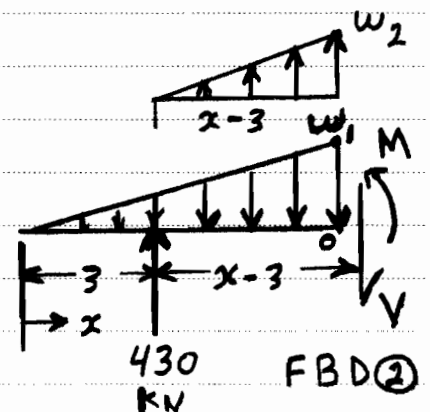
First, we need to determine R_A in FBD ①.

$$\sum M_B = 0 \Rightarrow 150(6) - 5R_A + 500(2.5) = 0 \Rightarrow R_A = 430 \text{ kN} \uparrow$$

Since the distributed load has to continue to the end once it starts, we need to make equivalent load as shown below.



Now, a section is made in the last segment (between A and B), and FBD ② is drawn.



Solution of HW # 15

$$\frac{w_1}{x} = \frac{800/3}{8} \Rightarrow w_1 = \frac{100}{3} x$$

$$\frac{w_2}{x-3} = \frac{500/3}{5} \Rightarrow w_2 = \frac{100}{3} (x-3)$$

$$\uparrow \Sigma M_0 = 0 \Rightarrow -430(x-3) + \frac{100}{3} x \left(\frac{x}{2}\right) \left(\frac{x}{3}\right) - \frac{100}{3} (x-3) \left(\frac{x-3}{2}\right) \left(\frac{x-3}{3}\right) + M = 0$$

Moving terms to the other side and expressing them in a singularity function form:

$$M = -\frac{100}{18} \langle x-0 \rangle^3 + 430 \langle x-3 \rangle^1 + \frac{100}{18} \langle x-3 \rangle^3$$

$$EI \frac{dv}{dx} = \int M dx = -\frac{25}{18} \langle x-0 \rangle^4 + 215 \langle x-3 \rangle^2 + \frac{25}{18} \langle x-3 \rangle^4 + C_1$$

$$EI v = -\frac{5}{18} \langle x-0 \rangle^5 + \frac{215}{3} \langle x-3 \rangle^3 + \frac{5}{18} \langle x-3 \rangle^5 + C_1 x + C_2$$

$$B.C.s.: v_A = v(3) = 0 \quad \text{and} \quad v_B = v(8) = 0$$

Two equations and two constants: We can solve.

$$v(3) = 0 = -\frac{5}{18} (3)^5 + 3C_1 + C_2 \quad \textcircled{1}$$

$$v(8) = 0 = -\frac{5}{18} (8)^5 + \frac{215}{3} (5)^3 + \frac{5}{18} (5)^5 + 8C_1 + C_2 \quad \textcircled{2}$$

Solving eqs. ① and ② yields

$$C_1 = -\frac{475}{3} \quad \& \quad C_2 = \frac{1085}{2} \quad \Rightarrow$$

$$v = \left[-\frac{5}{18} \langle x-0 \rangle^5 + \frac{215}{3} \langle x-3 \rangle^3 + \frac{5}{18} \langle x-3 \rangle^5 - \frac{475}{3} x + \frac{1085}{2} \right] / EI$$

units in kN, m