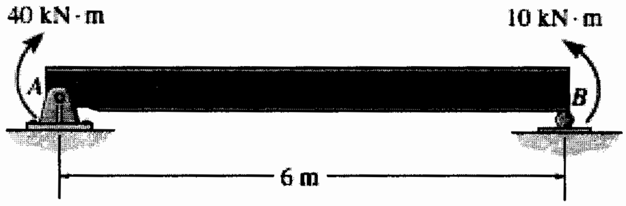
Solution of HW # 15

Problem # 1:

Given:
The beam shown
$E = 200\ GPa$
$I = 39.9 (10)^{-6}\ m^4$



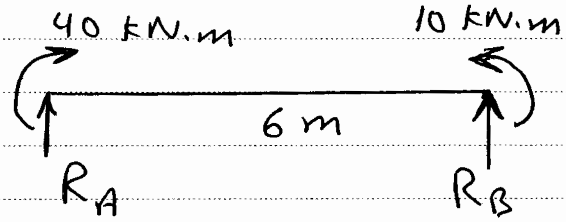
Required:
Maximum deflection

Solution:

We will start with the moment equation.
We may start with the load equation.

We need to find R_A. (Why ?!)

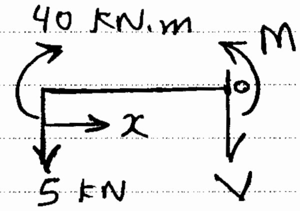
In FBD①,



$\overset{+}{\circlearrowleft} \Sigma M_B = 0 \Rightarrow -40 + 10 - 6R_A = 0$ FBD ①

$\Rightarrow R_A = -5\ kN = 5\ kN \downarrow$

To get the moment equation, we draw FBD② by taking a section "between A and B". Note that there is only one moment equation. (How ?!)



FBD②

$\overset{+}{\circlearrowleft} \Sigma M_o = 0 \Rightarrow M = 40 - 5x = EI \frac{d^2v}{dx^2}$ (kN.m)

$$\text{Slope} = \theta = \frac{dv}{dx} = \int \frac{M}{EI}\, dx \Rightarrow$$

$$EI \frac{dv}{dx} = \int (40 - 5x)\, dx = 40x - \frac{5}{2}x^2 + C_1$$

$$EIv = \int EI \frac{dv}{dx}\, dx = \int EI \left(40x - \frac{5}{2}x^2 + C_1\right) dx \Rightarrow$$

$$EIv = 20x^2 - \frac{5}{6}x^3 + C_1 x + C_2$$

Boundary Conditions (B.Cs):

We have 2 B.Cs. and 2 unknows (C_1 and C_2). Thus, we can find C_1 & C_2.

$v(0) = 0$ [That is the deflection is zero at $x = 0$ (A).] $\Rightarrow$

$0 = 0 + 0 + 0 + C_2 \Rightarrow C_2 = 0$

$v(6) = 0 \Rightarrow 0 = 20(6)^2 - \frac{5}{6}(6)^3 + C_1(6) \Rightarrow$
$C_1 = -90$

"with appropriate units"!

Thus,

$$EIv = 20x^2 - \frac{5}{6}x^3 - 90x$$

To get v_{max}, we set $\frac{dv}{dx} = 0 \Rightarrow$ get x. (Why?)

$\Rightarrow \frac{dv}{dx}EI = 40x - \frac{5}{2}x^2 - 90 \equiv 0 \Rightarrow$

$x^2 - 16x + 36 = 0 \Rightarrow x = 13.292$ or 2.7085 m

✗ (outside the range) ✓

$$\Rightarrow v_{max} = \left[20(2.7085)^2 - \frac{5}{6}(2.7085)^3 - 90(2.7085)\right/ 200(10)^9(39.9)(10)^{-6}\Big](10)^3$$

kN → N

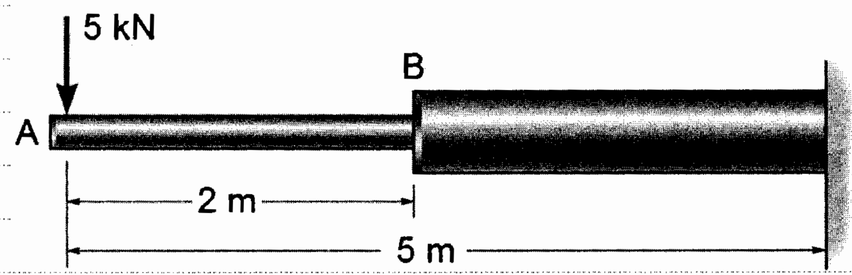
$\Rightarrow v = -0.014236$ m

$$\Rightarrow \boxed{v = 14.24 \text{ mm} \downarrow \text{ @ } x = 2.709 \text{ m}}$$

" Very small as in most applications "

Problem #2:

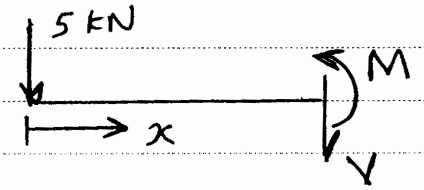
Given:
The beam shown
$E =$ Constant
$I = I_o$ for AB
$= 2 I_o$ for BC



Required:
Deflection at B in terms of EI_o

Solution:
We will start with the moment equation. Note that there is only one moment equation for both AB and BC; however, there is a "discontinuity" in I at B. Thus, we need to divide the domain into two parts/segments (from A to B and from B to C) and use the "concept of continuity" in slope and deflection at B. ⇒

From the FBD,



FBD

$$M = -5x \quad (\text{kN.m})$$

AB $(0 \le x \le 2)$:

$$EI_o \frac{d^2v}{dx^2} = M = -5x$$

$$EI_o \frac{dv}{dx} = \int M\,dx = -\frac{5}{2}x^2 + C_1 \qquad (1)$$

$$EI_o v = \int \left(-\frac{5}{2}x^2 + C_1\right) dx = -\frac{5}{6}x^3 + C_1 x + C_2 \qquad (2)$$

BC $(2 \le x \le 5)$:

$$2EI_o \frac{dv}{dx} = M = -5x \quad \Rightarrow$$

$$2EI_0 \frac{dv}{dx} = \int(-5x)\,dx = -\frac{5}{2}x^2 + C_3 \quad (3)$$

$$2EI_0 v = \int\left(-\frac{5}{2}x^2 + C_3\right)dx = -\frac{5}{6}x^3 + C_3 x + C_4 \quad (4)$$

B.C.s :

$v(5) = v_C = 0$ on range BC [eq (4)]

$\frac{dv}{dx}(5) = \theta_C = 0$ on range BC [eq. (3)]

Continuity Conditions (C.C.s.) :

$v_{AB} = v_{BC}$ @ B

$\left(\frac{dv}{dx}\right)_{AB} = \left(\frac{dv}{dx}\right)_{BC}$ @ B

Now, we have $\underset{\sim}{4}$ B.C.s & C.C.s. and $\underset{\sim}{4}$ unknown constants. We can find them.

$v(5) = 0$ in eq. (4), $\Rightarrow -\frac{5}{6}(5)^3 + 5C_3 + C_4 = 0 \quad (5)$

$\frac{dv}{dx}(5) = 0$ in eq. (3), $\Rightarrow -\frac{5}{2}(5)^2 + C_3 = 0 \Rightarrow$

$C_3 = 62.5 \Rightarrow$ into eq. (5) $\Rightarrow C_4 = -625/3$

$\left(\frac{dv}{dx}\right)(2)_{AB} = \left(\frac{dv}{dx}\right)(2)_{BC}$ [eq. (1) = eq. (3) @ B]

$$\Rightarrow \frac{1}{EI_0}\left[-\frac{5}{2}(2)^2 + C_1\right] = \frac{1}{2EI_0}\left[-\frac{5}{2}(2)^2 + 62.5\right]$$

$\Rightarrow C_1 = 36.25$

$v(2)_{AB} = v(2)_{BC}$ [eq. (2) = eq. (4) @ B] $\Rightarrow$

$$\frac{1}{EI_o}\left[-\frac{5}{6}(2)^3 + 36.25\,(2) + C_2\right] = \frac{1}{2EI_o}\left[-\frac{5}{6}(2)^3 + 62.5(2) - \frac{625}{3}\right]$$

$$\Rightarrow \quad C_2 = -665/6$$

$v_B = v(2)$ on AB or on BC

Use AB. $\Rightarrow$ Eq. 2 $\Rightarrow$

$$v_B = \left[-\frac{5}{6}(2)^3 + 36.25\,(2) - 665/6\right]/EI_o \Rightarrow$$

$$\boxed{v_B = -45/EI_o}$$

check using eq. 4 [BC] :

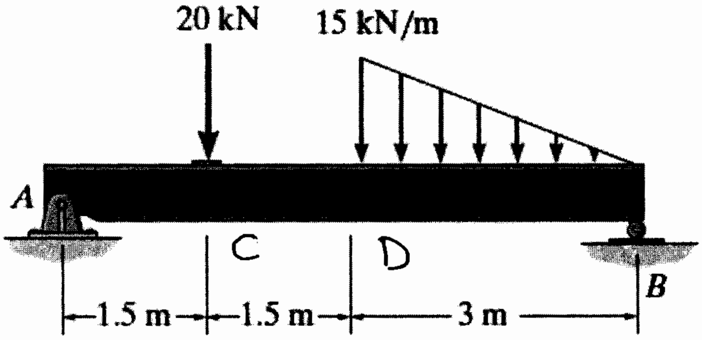
$$v_B = \left[-\frac{5}{6}(2)^3 + 62.5\,(2) - 625/3\right]/2EI_o \Rightarrow$$

$$v_B = -45\ /EI_o \qquad \text{ok !}$$

Note that if we only need v_B, we can use eq. 4 after finding C_3 and C_4 from the B.C.s. Thus, no need for the C.C.s. to find C_1 and C_2. In this case, we can not "check". However, if we need the eqs. of the elastic curves (def. eqs.), then we do as we did above.

Problem # 3:

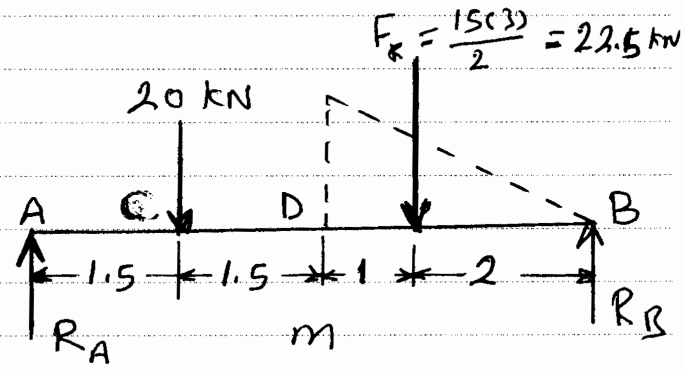
Given:
The beam shown
$E = 200$ GPa
$I = 65(10)^6$ mm^4



Required:
Maximum deflection

Solution:

Starting with the moment equation, we need R_A in FBD ①. ⇒



FBD ①

$\circlearrowleft + \sum M_B = 0 \Rightarrow$

$-6R_A + 20(4.5) + 22.5(2) = 0 \Rightarrow$

$R_A = 22.5$ kN

To get the moment equation, we make a section in the last segment and draw FBD ② for the left part.

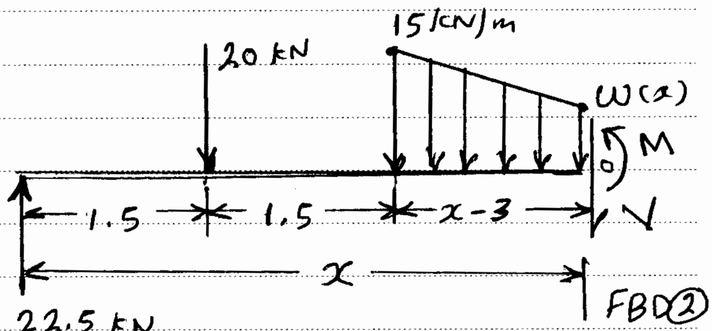
⇓ DB

$\frac{w(x)}{15} = \frac{6-x}{3} \Rightarrow$

$w(x) = 5(6-x)$

$F_1 = 15(x-3)$ ↓

$F_2 = (15-w)\left(\frac{x-3}{2}\right)$ ↑



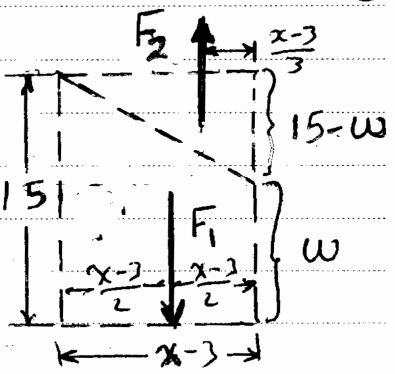
The locations of F_1 and F_2 are as shown.

$F_2 = (15-w)\left(\frac{x-3}{2}\right)$

$= [15 - 5(6-x)](x-3)/2$

$= 5[(3-6+x)](x-3)/2$

$= \frac{5}{2}(x-3)^2$



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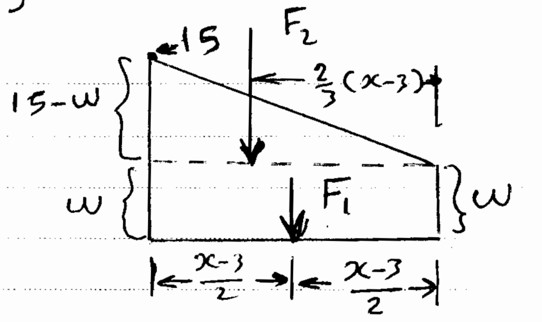
Try taking F_1 and F_2 as shown! $\Longrightarrow$

What did you find ?!



$+\circlearrowleft \; \Sigma M_o = 0 \Rightarrow$

$$M - 22.5x + 20(x-1.5) + 15(x-3)\left(\frac{x-3}{2}\right) - \frac{5}{2}(x-3)^2\left(\frac{x-3}{3}\right) = 0$$

$F_1 = w(x-3) \downarrow$

$F_2 = (15-w)(x-3)/2 \downarrow$

Moving the terms to the other side and using the singularity function form, we get :

$$M = 22.5\langle x-0\rangle^1 - 20\langle x-1.5\rangle^1 - \frac{15}{2}\langle x-3\rangle^2 + \frac{5}{6}\langle x-3\rangle^3$$

$$EI\frac{d^2v}{dx^2} = M \Rightarrow EI\frac{dv}{dx} = \int M\,dx \Rightarrow$$

$$EI\frac{dv}{dx} = 11.25\langle x-0\rangle^2 - 10\langle x-1.5\rangle^2 - \frac{15}{6}\langle x-3\rangle^3 + \frac{5}{24}\langle x-3\rangle^4 + C_1$$

$$EIv = 3.75\langle x-0\rangle^3 - \frac{10}{3}\langle x-1.5\rangle^3 - \frac{15}{24}\langle x-3\rangle^4 + \frac{1}{24}\langle x-3\rangle^5 + C_1x + C_2$$

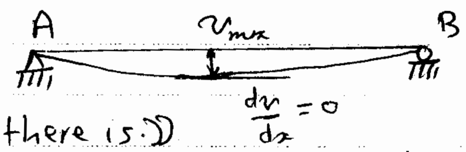
B.C.s. : $v(0) = 0 \Rightarrow 0 - 0 - 0 + 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$

Note that the value within $\langle x-a\rangle$ can never be negative. If so, it is ZERO! (why and how ?!)

$v(6) = v_B = 0 \Rightarrow 3.75(6)^3 - \frac{10}{3}(4.5)^3 - \frac{15}{24}(3)^4 + \frac{1}{24}(3)^5 + 6C_1 = 0$

$\Rightarrow C_1 = -77.625$ (units in kN, m)

v_{max} is @ $\frac{dv}{dx} = 0$



(OR at the free end if there is.)

$\Rightarrow$ Set $\frac{dv}{dx} = 0 \Rightarrow 11.25x^2 - 10\langle x-1.5\rangle^2 - \frac{15}{6}\langle x-3\rangle^3 + \frac{5}{24}\langle x-3\rangle^4 - 77.625 = 0$

We can either find the four roots of this 4^{th} degree equation, or assume the root within our range $(0 \leq x \leq 6)$

is between 0 and 3 m (as there is only one possible root within this range, as seen in the figure above). In this case, we have to check our assumption ($x \overset{?}{\leq} 3$). This will simplify our solution as the 3rd and 4th order terms are dropped. (How?!) $\Rightarrow$

$$11.25\,x^2 - 10\,(x-1.5)^2 - 77.625 = 0 \Rightarrow$$

$$x = 2.96997 \quad \underline{\text{or}} \quad -26.96997$$

« ok » « not possible »

$\Rightarrow$ $x_m = 2.970$ m « within our range 0–3 » $\Rightarrow$ $\underline{\text{ok}}$

Thus v_{max} @ $x_m = 2.970$ m $\Rightarrow$

$$v_{max} = \left[3.75\,x_m^3 - \frac{10}{3}(x_m - 1.5)^3 - 77.625\,x_m\right](10)^3 \Big/ 200(10)^9\,65(10)^{-6}$$

$$\Rightarrow \quad \boxed{v_{max} = -0.010992 \text{ m} = 10.99 \text{ mm} \downarrow}$$

Again, very small compared with $L = 6$ m

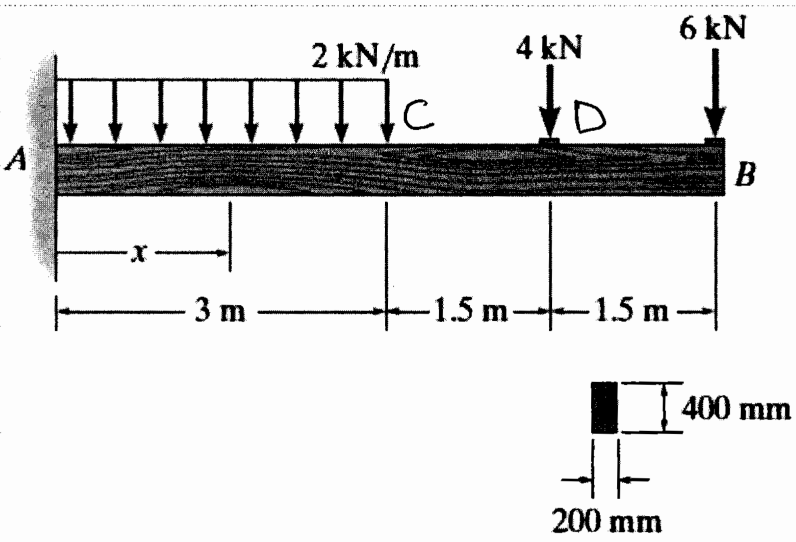
Solution of HW # 15

Problem # 4:

Given:

The beam shown

$E = 12\ GPa$



Required:

The eq. of the elastic curve (deflection)

Deflection & slope @ B

Solution:

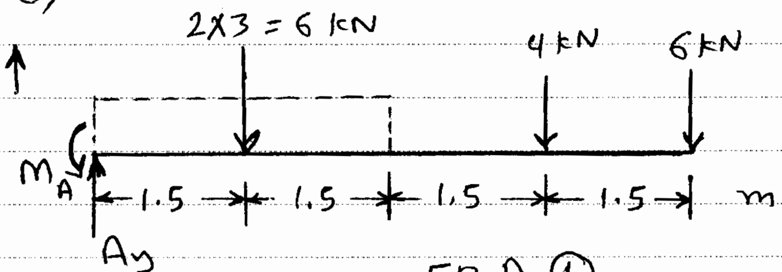
First, we need to find the reactions. From FBD①,

$\uparrow \Sigma F_y = 0 \Rightarrow A_y = 16\ kN \uparrow$

$\circlearrowleft \Sigma M_A = 0 \Rightarrow$

$M_A - 6(1.5) - 4(4.5) - 6(6) = 0$

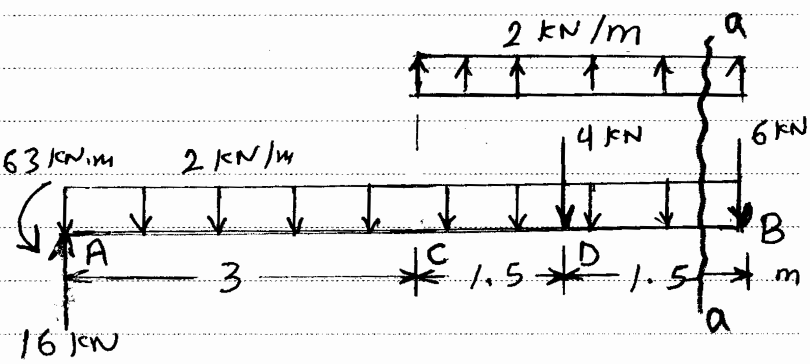
$\Rightarrow M_A = 63\ kN.m$ ↺



FBD ①

Using the singularity function, you need to remember that once the distributed L starts (from anywhere/x), it has to go all the way up to the end. (Why ?!) Thus, the 2 kN/m - load has to go from A to B. Therefore, we have to add an upward load equals to 2 kN/m from C to B, thus making the total loads equal to the original loading. This is shown in the figure below.

Now, we make a section (a-a) in the last segment (DB), and draw a FBD by taking



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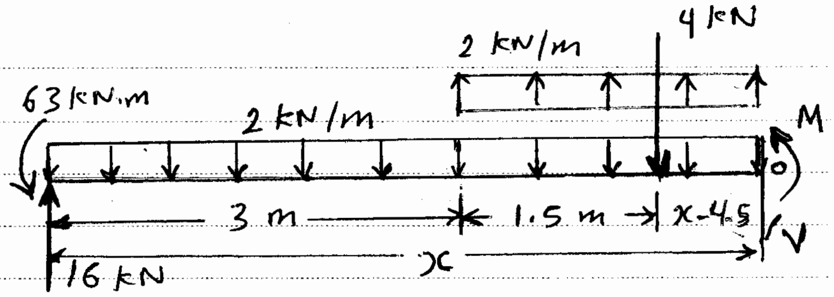
the lef part, as shown FBD ②.

Note that any concentrated load / moment (whether applied or reaction) does not appear! Does it mean "it does not affect the beam" ?! NO !! HOW ??!!



FBD ②

$\circlearrowleft + \sum M_o = 0 \Rightarrow$ In a singularity function form,

$$M = 16\langle x-0\rangle^1 - 63\langle x-0\rangle^0 - \frac{2}{2}\langle x-0\rangle^2 + \frac{2}{2}\langle x-3\rangle^2 - 4\langle x-4.5\rangle^1$$

$$EI\frac{dv}{dx} = 8\langle x-0\rangle^2 - 63\langle x-0\rangle^1 - \frac{1}{3}\langle x-0\rangle^3 + \frac{1}{3}\langle x-3\rangle^3 - 2\langle x-4.5\rangle^2 + C_1$$

$$EI\,v = \frac{8}{3}\langle x-0\rangle^3 - \frac{63}{2}\langle x-0\rangle^2 - \frac{1}{12}\langle x-0\rangle^4 + \frac{1}{12}\langle x-3\rangle^4 - \frac{2}{3}\langle x-4.5\rangle^3 + C_1 x + C_2$$

B.C.s. : dv/dx and $v = 0$ @ $x = 0$ (A) $\Rightarrow$

$dv/dx\,(0) = 0 - 0 - 0 + 0 - 0 + C_1 = 0 \Rightarrow C_1 = 0$

$v(0) = 0 = 0 - 0 - 0 + 0 - 0 + 0 + C_2 \Rightarrow C_2 = 0$

Thus,

$$\boxed{v = \frac{1}{EI}\left[\frac{8}{3}\langle x-0\rangle^3 - \frac{63}{2}\langle x-0\rangle^2 - \frac{1}{12}\langle x-0\rangle^4 + \frac{1}{12}\langle x-3\rangle^4 - \frac{2}{3}\langle x-4.5\rangle^3\right.}$$

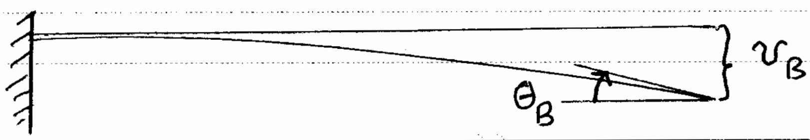
At B, $x = 6\,m$; $E = 12(10)^9\ N/m^2$

$I = bh^3/12 \Rightarrow I = (0.2)(0.4)^3/12 = \frac{3.2}{3}(10)^{-3}\ m^4$

slope $= \theta_B = dv/dx = [8(6)^2 - 63(6) - \frac{1}{3}(6)^3 + \frac{1}{3}(3)^3 - 2(1.5)^2](10)^3/EI$

$\Rightarrow \boxed{\theta_B = -0.01230\ rad = -0.7050^\circ}$ « cw »

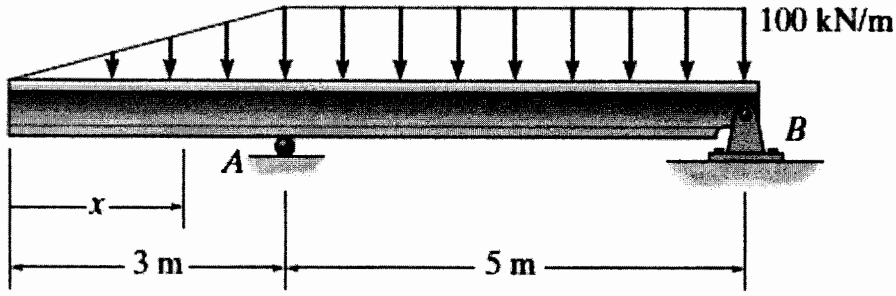
$v_B = [\frac{8}{3}(6)^3 - \frac{63}{2}(6)^2 - \frac{1}{12}(6)^4 + \frac{1}{12}(3)^4 - \frac{2}{3}(1.5)^3]/EI \Rightarrow \boxed{v_B = -0.05168\ m \downarrow}$



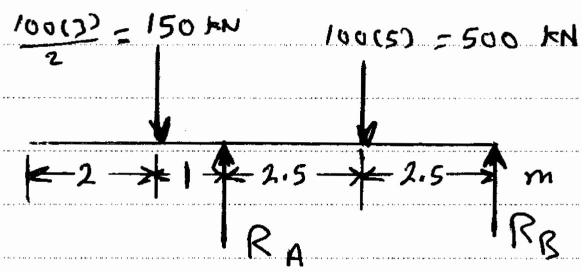
not to scale

Problem # 5:

Given:
The beam shown
EI = Constant



Required:
Eq. of elastic curve (deflection)

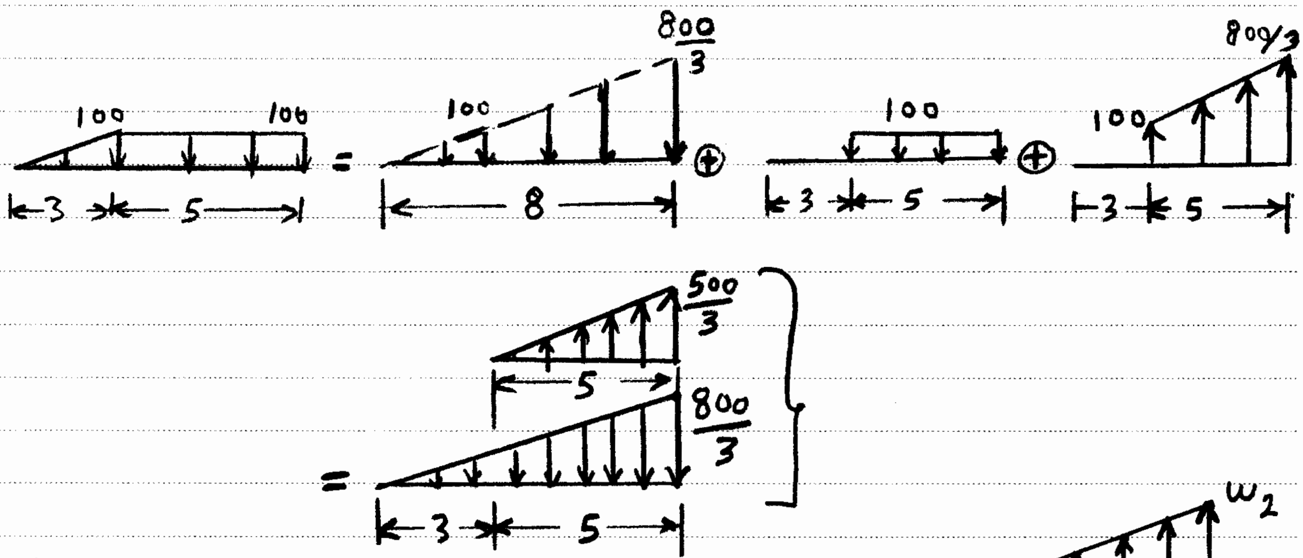


Solution:
First, we need to determine R_A in FBD ①.

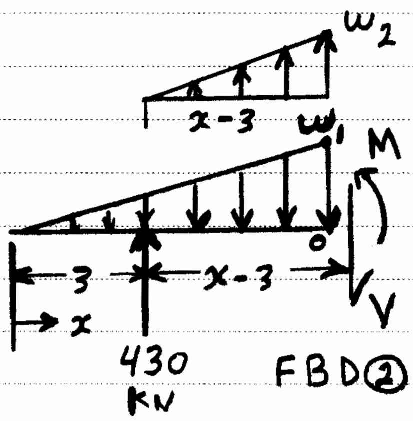
FBD ①

$\overset{+}{\curvearrowleft} \sum M_B = 0 \Rightarrow 150(6) - 5R_A + 500(2.5) = 0 \Rightarrow R_A = 430 \text{ kN} \uparrow$

Since the distributed load has to continue to the end once it starts, we need to make equivalent load as shown below.



Now, a section is made in the last segment (between A and B), and FBD ② is drawn.



FBD ②

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$$\frac{w_1}{x} = \frac{800/3}{8} \Rightarrow w_1 = \frac{100}{3}x$$

$$\frac{w_2}{x-3} = \frac{500/3}{5} \Rightarrow w_2 = \frac{100}{3}(x-3)$$

$$\circlearrowleft + \sum M_o = 0 \Rightarrow -430(x-3) + \frac{100}{3}x\left(\frac{x}{2}\right)\left(\frac{x}{3}\right) - \frac{100}{3}(x-3)\left(\frac{x-3}{2}\right)\left(\frac{x-3}{3}\right) + M = 0$$

Moving terms to the other side and expressing them in a singularity function form:

$$M = -\frac{100}{18}\langle x-0\rangle^3 + 430\langle x-3\rangle^1 + \frac{100}{18}\langle x-3\rangle^3$$

$$EI\frac{dv}{dx} = \int M dx = -\frac{25}{18}\langle x-0\rangle^4 + 215\langle x-3\rangle^2 + \frac{25}{18}\langle x-3\rangle^4 + C_1$$

$$EI\,v = -\frac{5}{18}\langle x-0\rangle^5 + \frac{215}{3}\langle x-3\rangle^3 + \frac{5}{18}\langle x-3\rangle^5 + C_1x + C_2$$

B.C's.: $v_A = v(3) = 0$ and $v_B = v(8) = 0$

Two equations and two constants: We can solve.

$$v(3) = 0 = -\frac{5}{18}(3)^5 + 3C_1 + C_2 \quad (1)$$

$$v(8) = 0 = -\frac{5}{18}(8)^5 + \frac{215}{3}(5)^3 + \frac{5}{18}(5)^5 + 8C_1 + C_2 \quad (2)$$

Solving eqs. (1) and (2) yields

$$C_1 = -\frac{475}{3} \quad \& \quad C_2 = \frac{1085}{2} \Rightarrow$$

$$\boxed{v = \left[-\frac{5}{18}\langle x-0\rangle^5 + \frac{215}{3}\langle x-3\rangle^3 + \frac{5}{18}\langle x-3\rangle^5 - \frac{475}{3}x + \frac{1085}{2}\right] \Big/ EI}$$

units in kN, m