1. Use the method of the Gauss Elimination and the method of the Cholesky decomposition to solve textbook Problem 11.6 page 313. Comments on the use of the two methods for the solution of the system of matrix equations Ax=b.

Solution:

$$\begin{bmatrix} 8 & 20 & 15 \\ 20 & 80 & 50 \\ 15 & 50 & 60 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 250 \\ 100 \end{Bmatrix};$$

a. First solution by the **Gauss Elimination** method;

$$\left[\begin{array}{ccc|c} 8 & 20 & 15 & 100 \\ 20 & 80 & 50 & 250 \\ 15 & 50 & 60 & 100 \end{array}\right]; Times\ R1\ by \frac{20}{8} and\ sub.from\ R2\ ,$$

$$also\ \ Times\ R1\ by \frac{15}{8} and\ sub.from\ R3\ to\ remove\ x_1\ from\ the\ second\ and\ third\ eqs.;$$

$$\left[\begin{array}{ccc|c} 8 & 20 & 15 & 100 \\ 0 & 30 & 12.5 & 0 \\ 0 & 12.5 & 31.875 & -87.5 \end{array}\right];$$

$$then\ times\ second\ row\ by \frac{12.5}{30} and\ sub.from\ R3\ to\ remove\ x_2 from\ eq3.;$$

$$\left[\begin{array}{ccc|c} 8 & 20 & 15 & 100 \\ 0 & 30 & 12.5 & 0 \\ 0 & 0 & 26.667 & -87.5 \end{array}\right];$$

Now we can do back substitution; $\boldsymbol{x_3 = -3.28}$; $\boldsymbol{x_2 = 1.37}$; $\boldsymbol{x_1 = 15.29}$ $Ans.$

b. Using the **Cholesky Decomposition;**

$$L_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} L_{ij} L_{kj}}{L_{ii}}; \quad i = 1,2,3 \dots \dots, k-1$$

$$L_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}; \quad i = 1,2,3 \dots \dots, k-1$$

$\Rightarrow l_{11} = 2\sqrt{2}$; $l_{21} = \frac{10}{\sqrt{2}}$; $l_{22} = \sqrt{30}$; $l_{31} = \frac{15}{2\sqrt{2}}$; $l_{32} = \frac{25}{2\sqrt{30}}$

$\therefore l_{33} = \sqrt{\frac{80}{3}}$ $\Rightarrow [L] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ \frac{10}{\sqrt{2}} & \sqrt{30} & 0 \\ \frac{15}{2\sqrt{2}} & \frac{25}{2\sqrt{30}} & \sqrt{\frac{80}{3}} \end{bmatrix}$ $\therefore [L][L]^T X = b$

Note that $[L]^T X$ in D $\Rightarrow [L][D] = b$; By solving there

$$\boldsymbol{D1 = \frac{50}{\sqrt{2}}}; \quad \boldsymbol{D2 = 0}; \ \boldsymbol{D3 = -16.94}$$

$[U]\{X\}=\{D\}; \{D\} = \begin{Bmatrix} 50/\sqrt{2} \\ 0 \\ -16.94 \end{Bmatrix}$

$$\Rightarrow \begin{bmatrix} 2\sqrt{2} & \frac{10}{\sqrt{2}} & \frac{15}{2\sqrt{2}} \\ 0 & \sqrt{30} & \frac{25}{2\sqrt{30}} \\ 0 & 0 & \sqrt{\frac{80}{30}} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} :$$

Carrying out the back subs.; $\mathbf{x_1 = 15.3}$; $\mathbf{x_2 = 1.4}$; $\mathbf{x_3 = -3.279}$

2. Use Gauss-Seidel method to solve Problem 11.11 textbook page 313 and ensure the iterative steps are stopped when the %-relative error is not more than ε_s=4%.

Solution:-

Problem11.11, Page 313;

$$10x_1 + 2x_2 - x_3 = 27 \qquad (1)$$

$$-3x_1 - 6x_2 + 2x_3 = -61.5 \qquad (2)$$

$$x_1 + x_2 + 5x_3 = -21.5 \qquad (3)$$

$$From\ \boldsymbol{Eq.1};\ x_1 = \frac{27 - 2x_2 + x_3}{10}\ ;Put \quad x_2 = 0\ and\ \ x_3 = 0; \ \rightarrow \quad \boldsymbol{x_1 = 2.7}$$

$$From\ \boldsymbol{Eq.2};\ x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6}\ ;Put \quad x_1 = 2.7\ and\ \ x_3 = 0; \ \rightarrow \quad \boldsymbol{x_2 = 8.9}$$

$$From\ \boldsymbol{Eq.3};\ x_3 = \frac{-21.5 - x_1 - x_2}{5}\ ;Put \quad x_1 = 2.7\ and\ \ x_2 = 8.9; \ \rightarrow \quad \boldsymbol{x_3 = -6.62}$$

Repeat this procedure until the relative error is < 4%; as it is shown in the following table:

iter #	x_1	x_2	x_3	ε_{s1} %	ε_{s2} %	ε_{s3} %
1	2.7	8.9	-6.62	100	100	100
2	0.258	7.914333	-5.93447	90.44444	11.07491	10.35549
3	0.523687	8.010001	-6.00674	102.9793	1.208791	1.217816
4	0.497326	7.999091	-5.99928	5.033667	0.136204	0.124096
5	**0.500253**	**8.000112**	**-6.00007**	0.588629	0.012764	0.013163

The solution is $\boldsymbol{x_1 = 0.500253}$; $\boldsymbol{x_2 = 8.000112}$; $\boldsymbol{x_3 = -6.00007}$