5.3 Stiff and Servo-Controlled Testing Machine

Relative stiffness: stiffness of rock / stiffness of machine

In soil mechanics testing, soils are generally much less stiff than the testing machine parts.

But in rock testing, the strain that builds up in the platens may be greater than those in the rock.

If the rock stiffness up on unloading is greater than that of the testing machine, the machine will "unstretch" more than the rock resulting in violent disintegration of the specimen.

"Soft machine" : machine with stiffness < stiffness of rock.

Machine (Figure drawing)

Rock

Loading to P_1 to $P_{failure}$, unloading to P_2 (Jump)

In Goodman's text
pp. 77

If $k_R > k_m \Rightarrow$ post peak

behavior can't be recorded.

* Remedies for a "soft" machine

i) reduce stiffness of rock: $\Delta = \frac{PL}{AE}$

by decreasing rock cross-sectional area. Stiffness $= \frac{P}{\Delta} = \frac{AE}{L}$

ii) load stiff steel bars in parallel with rock.

iii) Can servo-control machine sense force and displacements in rock and machine, feeds information back into machine to more loading platen backwards.

(stress controlled machine) more cost than (strain control machine)

5.4 <u>Brazilian Split Cylinder Test</u>: (used for super collider/conductor project)

$$s_t = \frac{2P}{pdt}$$

From elasticity theory:

$$s_r = +\frac{2P}{p}\frac{Cos\, q}{r}$$

$$s_q = t_{rq} = 0$$

- prove by plugging into equilibrium equations:

$$\frac{\partial s_r}{\partial_r} + \frac{1}{r}\frac{\partial t_{rq}}{\partial_q} + \frac{s_r - s_q}{r} = 0$$

$$\frac{1}{r}\frac{\partial s_q}{\partial_q} + \frac{\partial t_{rq}}{\partial_r} + \frac{2\, t_{rq}}{r} \; 0$$

Since d.cos θ = r

$$s_r = \frac{2P}{pd} \qquad \text{@ any point in the circle}$$

This is for half space, but if the circle becomes a free surface (not tractions), the compressive force due to P must be balanced by tension forces. (see Timoshinko & Goodier “Theory of Elasticity” Section 4.1)

We can’t use point load for soft rock.

⇒ ASTM specifies a radius for the frame for every core diameter.

* Note tensile strength is more than that from direct tensile test since crack opens more in direct tension
⇒ strength reduced.

* This is the critical location where failure is initiated. Ratio of compressive to tensile stress is 3:1.

* Mohr’s circle for Brazilian test.

5.5 Flexural test:

$$M_{\max} = {}^{P}\!/_{2} \cdot {}^{L}\!/_{3} = \frac{PL}{6}$$

$$s_t = \frac{M_{\max}\, c}{I} \quad , \qquad c = \frac{d}{2}$$

$$I = \frac{pd^4}{64}$$

$$s_t = \frac{\frac{PL}{6}\cdot\frac{d}{2}}{\frac{pd^4}{64}} = \frac{16}{3}\,\frac{PL}{pd^3}$$

$$s_t = \frac{16}{3}\,\frac{PL}{pd^3}$$ where P = load at failure

5.6 Ring Shear Test

Direct shear test of rock

$$t_{\max} = \frac{P}{2\,p\left(\frac{d}{2}\right)^2} = \frac{2P}{pd^2}$$

area

two planes (double shear)

$$t_{\max} = \frac{2P}{pd^2}$$

6.0 Stress-strain Behavior of Intact Rock

6.1 Stress

Scalar (temperature)
↓
Vector : 3 components (load)
↓
Tensor : 9 components

o Total stress tensor: $$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \tilde{\sigma}_{ij}$$

$\sigma = \sigma_h + \sigma_d$ total stress = hydrostatic + deviatoric

total hydrostatic deviatoric

a. hydrostatic component (non-deviatoric, mean stress, dilational, isotropic, spherical):

$$\sigma_h = \begin{bmatrix} \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3} & 0 & 0 \\ 0 & \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3} & 0 \\ 0 & 0 & \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3} \end{bmatrix} = \frac{1}{3}\tilde{\sigma}_{kk}\,\delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$
$= 0$ if $I \neq j$

b. deviatoric stress component

$$\sigma_d = \begin{bmatrix} \left(\sigma_{11} - \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}\right) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \left(\sigma_{22} - \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}\right) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \left(\sigma_{33} - \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}\right) \end{bmatrix} = \tilde{\sigma}_{ij} - \tfrac{1}{3}\tilde{\sigma}_{kk}\,\delta_{ij}$$

<u>Example</u>: triaxial test

$$s_{ij} = \begin{bmatrix} \frac{P}{A} & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(\frac{P}{A} + 2P) & 0 & 0 \\ 0 & \frac{1}{3}(\frac{P}{A} + 2P) & 0 \\ 0 & 0 & \frac{1}{3}(\frac{P}{A} + 2P) \end{bmatrix} + \begin{bmatrix} \frac{2}{3}(\frac{P}{A} - P) & 0 & 0 \\ 0 & \frac{1}{3}(P - \frac{P}{A}) & 0 \\ 0 & 0 & \frac{1}{3}(P - \frac{P}{A}) \end{bmatrix}$$

* Read pp 55-78, 179-187

* Do problems #1, #4, #5, pp 218 - HW #2

Exam: - Closed book
- Till today
- ≈ 40 min

o $\sigma = \sigma_h + \sigma_d$

Why we break it see fig. pp. 69 text.

$$\frac{\Delta v}{v_o} = e_{11} + e_{22} + e_{33}$$

$$s_{mean} = \frac{s_{11} + s_{22} + s_{33}}{3}$$

1. Concave up : strain hardening
because micro cracks are closing (dilation)
⇒ plastic permanent deformation with some rebound.

2. Elastic

Elastic grain deformation | Pore-deformation
Grain compression

3. Cracking : pore structure collapse

4. : No peak load response

Fig. 3.7

Significance of deviatoric vs. non-deviatoric loading:

*. Deviatoric stress produces distortion and destruction, while non-deviatoric stress generally doesn't.

**. Deviatoric loading is characterized by a "peak load"

6.2 Strain, Volumetric strain and Dilation

Strain tensor:

$$\begin{bmatrix} e_{11} & g_{12} & g_{13} \\ g_{21} & e_{22} & g_{23} \\ g_{31} & g_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

By definition, volumetric strain = $\frac{\Delta V}{V_o} = e_{kk}$

In triaxial test $e_{axial} = e_{11} = \frac{\Delta L}{L}$

$$e_{lateral} = e_{22} = e_{33} = \frac{\Delta d}{d}$$

$$\frac{\Delta V}{V_o} = e_{axial} + 2\, e_{lateral}$$

Since n = Poisson Ratio = - $\frac{e_{lateral}}{e_{axial}}$

$$\therefore \frac{\Delta V}{V_o} = e_{axial}\ (1 - 2n)$$

V_o & ε_{axial} can be measured
Measure ΔV : vol. of water in/out of chamber
Then calculate ν.

Fig. 4.2.7 course pack pp19.

When sample star expanding (Dilating)

$\nu > .5$?? violates elasticity theory.

6.3 Elasticity

$\varepsilon_{ij} = \{ \quad \} \ \sigma_{ij}$ Constitutive equation

(1) $\varepsilon_I = \{9 \times 9\} \ \sigma_I$ $i = 1,2,3, \ldots 9$

81 elastic constants would be needed to fully describe the relationship between σ & ε.

Are all 81 constants unique (different)?

(2) By symmetry. $\varepsilon_{xy} = \varepsilon_{yx}$ and $\sigma_{xy} = \sigma_{yx}$ …..

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{x3} \\ e_{yx} & e_{yy} & e_{y3} \\ e_{3x} & e_{3y} & e_{2} \end{bmatrix} \quad \& \quad \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{3x} & s_{3y} & s_{3z} \end{bmatrix}$$

$\therefore$ we recognize that only 6 strain and 6 stress tensors are unique. Number of constants reduces to 36.

(3) Furthermore, it may be shown that a relationship between σ_{yy} and ε_{xx} is identical to that between σ_{xx} and ε_{yy}.
[from strain energy consideration, receproty theory]

$\Rightarrow$ This reduces our # of constants to 36-15 = 21, for a totally anisotropic material.

(4) If there are 3 mutually perpendicular directions of symmetry, the material is said to be <u>orthotropic</u>. The number of constants is reduced to <u>9</u>. [Wood is a good example]

E_x, E_y, E_z, ν_{yx}, ν_{zx}, ν_{zy}, G_{xy}, G_{yz}, G_{zx} .

(5) If we can assume that a material is "transversely isotropic" the # of constants is reduced to 5. [stratified soil]/

Note: text used.

$E_x = E_y$

$\nu_{zx} = \nu_{zy}$

$G_{yz} = G_{xz}$

$$G_{yx} = \frac{E_x}{2(1+?_{yx})}$$

(6) If a material is totally isotropic: $\Rightarrow$ only 2 constants

$E_x = E_y = E_z$

$\nu_{3y} = \nu_{3y} = \nu_{xy}$

$G_{xy} = G_{xz} = G_{yz}$

$$K = \frac{E}{3(1-2?)} = \frac{s_{main}}{\frac{?V}{V_o}} = \frac{\frac{s_{11}+s_{22}+s_{33}}{3}}{e_{11}+e_{22}+e_{33}}$$

81→36→21→9→5→2

Wood — Layered Clays — Clay — Rock or clay

text ⇒ Eq, 6,7 Eq. 6.9 Eq 6.1

6.4 Some basic definition pertaining to stress-strain behavior:

1. Linearly Elastic material: $\sigma = E\varepsilon$

2. Perfectly Elastic: $\sigma = f(\varepsilon)$

E_0, E_s, E_t (L, U, R)

3. Linear (but not elastic)

4. Elastic

5. Permanent Set or
Permanent Deformation = δ

$\delta = f(\varepsilon)$

ε — strain level

6. Ductile state: rock sustains permanent deformations without loosing its ability to resist load.

7. Brittle sate: condition in which the ability to resist load decrease with increasing deformation.

8. Uniaxial compressive strength: makes the transition from ductile to brittle behavior.

9. Brittle to ductile transition pressure: the confining pressure at which a rock will exhibit no brittle behavior.

See Fig. 3.9 / pp.21 pack.

Ch. 7.0 Failure Theories for Rocks

- Coulomb (1773)
- Mohr (1900)
- Griffith (1921)
- Modified Griffith (1962)
- Empirical Criteria

7.1 Coulomb – straight line theory

$| \tau | = \sigma_n \tan \phi + S_i$

τ : shear stress across a plane at a point at which Strength failure occurs

σ_n : normal stress on the plane on which failure occurs

ϕ : angle of shearing resistance

S_i : shear axis intercept

7.2 Mohr : general (not linear)

$| \tau | = f(\sigma_n)$

Failure line doesn’t have to be a straight line.
In fact, it is commonly concave downward.

- Coulomb criteria is a special case of Mohr criteria.

- Mohr envelop is a tangent to Mohr circle.

- Mohr envelop can be plotted as a result of drawing tangent to Mohr's circles from different tests.

  - may not fit, different tests

- Common relationships

$$\sigma_{1,f} = q_u + \sigma_3 \tan^2 (45 + \phi/2)$$

$$q_u = 2\, S_i \tan (45 + \phi/2)$$

(See Appendix 4 (text) for derivation)

$$\frac{S_i}{q_u} = \frac{1}{2} \text{ Cot } (45 + \phi/2)$$ see Fig. 1.12.

Fig. 1.11. Elastic modulus of various rock grades as a function of the uniaxial compressive strength (after Kikuchi et al. 1982)

Fig. 1.12. Cohesion value of various rock grades as a function of the uniaxial compressive strength (after Kikuchi et al. 1982).

Max. stress ratio can also be determined:

$$\sigma_{1f} = q_u + \sigma_3 \tan^2 (45 + \phi/2)$$

$$1 = \frac{q_u}{\sigma_{1f}} + \frac{\sigma_3}{\sigma_{1f}} \tan^2 (45 + \phi/2)$$

$$1 = \frac{q_u}{q_{1f}} + K \tan^2 (45 + \phi/2)$$

$$K_{max} = \left(1 - \frac{q_u}{\sigma_{1f}}\right) \cot^2 (45 + \phi/2)$$

Fig. 1.11 (1982)

This is known 1966 see Fig. 1.4 – igneous (course pack
1.5 – sedimentary
1.6 – metamorphic

Classification system.

Ex. B M

high strength average modulus ratio

7.3 Griffith Brittle Crack Theory (1921)

- take into account cracks (based on presence of tiny imperfection everywhere)
- failure by continuation of cracks
- failure by tension at crack tip

He hypothesized that fractures caused by stress concentrations at the tips of minute cracks which pervade the material. Fracture is initiated when the max. stress near the tip of the most favorably oriented crack reaches a value which is a characteristic of the material.

$\sigma_1 > \sigma_3$

* Assume cracks are elliptical.

1. σ_1 , σ_3
2. σ_a , σ_b
3. σ_t
4. Cartezian coord. $\rightarrow$ elliptical coordinate

   X , Y R , T

   T = angle

   X =

   Y =

   at crack tip R = R_o << 1 T = 0

@ crack tip $\quad R = R_0 \ll 1$

$T = 0$

$x = c \sin h\, R \sin T$

$y = c \cos h\, R \cos T$

Note: if $T = 0$ and $R \ll 1.0$

$x = 0$

$y = c$ (i.e. ½ crack length)

5. At small R_o $\rightarrow$ σ_t (3.8)

6. Max & min. σ_t $\Rightarrow$ (3.10)

7. Plug 3.5 & 3.6 into 3.9 $\Rightarrow$ (3.10)

8. Find $\theta_{critical}$ to maximize σ_t $\Rightarrow$ $\dfrac{\partial \sigma_t}{\partial \theta}$ $\Rightarrow$ (3.11)

9. $\theta = 90^o$ into (3.10) $\Rightarrow$ (3.13)

$$s_{Tf} = \frac{1}{8}\frac{(s_1 - s_3)^2}{(s_1 + s_3)} \qquad \text{Griffith Criteria} \qquad (3.14)$$

From 3.14

$(\sigma_1 - \sigma_3)^2 = 8\, T_o\, (\sigma_1 + \sigma_3)$ when $\sigma_1 + 3\,\sigma_3 > 0$ (1.a)

$\sigma_3 = -\,T_o$ when $\sigma_1 + 3\,\sigma_3 < 0$ (1.b)

- look at a representation in σ_1-σ_3 plane:

- look at a representation in p-q plane:

Let $p = ½ (\sigma_1 + \sigma_3)$

$q = ½ (\sigma_1 - \sigma_3)$

Then Eqn. (1.a) may be written as

$(2q)^2 = 8 T_o 2p$ when $4p - 2q > 0$

or $q^2 = 4 T_o p$ when $2p > q$ (2.a)

And eqn. (1.b) becomes

$p\text{-}q = - T_o$

$q = p + T_o$ when $2p < q$ (2.b)

$(\sigma_1 - \sigma_3)^2 = 8\, T_o\, (\sigma_1 + \sigma_3)$ for $\sigma_1 + 3\,\sigma_3 > 0$

$\sigma_3 = -\, T_o$ for $\sigma_1 + 3\,\sigma_3 < 0$

$$p = \frac{1}{2}(\sigma_1 + \sigma_3)$$

$$q = \frac{1}{2}(\sigma_1 - \sigma_3)$$

<u>Griffith Criteria</u>:

for $2\,p > q$ $\quad q^2 = 4\, T_o\, p$

for $2\,p < q$ $\quad q = p + T_o$

<u>Note</u>: in a p-q diagram, the line failure envelope connects the tops of Mohr circles.

<u>Note</u>: The Mohr's circles for all points of the line AB touch at point A

- To find the Mohr envelope for points on the parabola BC, we have to find the envelope of the circles with center (p,o) and radius q. The equation for such a curve is:

$$f(p) = (\sigma - p)^2 + \tau^2 = q^2$$

$$(\sigma - p)^2 + \tau^2 - 4\, T_o\, p = 0 \qquad \text{------} \quad (3)$$

Since $\dfrac{\partial f}{\partial p} = 0$ (i.e. zero slope) at the <u>top</u> of circle

Then from (3) $\Rightarrow$ $2\,(\sigma - p)\,(-1) + 0 - 4\, T_o = 0$

$$-\,2\,(\sigma - p) - 4\, T_o = 0$$

$$-\,2\,\sigma + 2\,p - 4\, T_o = 0$$

$$- \sigma + p - 2\, T_o = 0$$

$$p = \sigma + 2\, T_o \qquad \text{------} \quad (4)$$

Combining (3) & (4),

$$[\sigma - (\sigma + 2\, T_o)]^2 + \tau^2 - 4\, T_0\, (\sigma + 2\, T_o) = 0$$

$$4\, T_o^2 + \tau^2 - 4\, T_o\, \sigma - 8\, T_o^2 = 0$$

$$\tau^2 = 4\, T_o\, (\sigma + T_o) \qquad \text{------} \quad (5)$$

This is Griffith theory for brittle crack.

shear stress = f (normal, another term)

σ	τ
0	$2.0\, T_o$
$2\, T_o$	$2.8\, T_0$
$3\, T_o$	$4.0\, T_o$

<u>For unconfining comp. stress</u>

$\sigma_3 = 0$

$(\sigma_1 - \sigma_3)^2 = 8\, T_o\, (\sigma_1 + \sigma_3)$

$\sigma_1^2 = 8\, T_o\, \sigma_1 \quad \Rightarrow \quad \sigma_1 = 8\, T_o$ or $q_u = 8\, T_o$

$\frac{q_u}{T_o} = 8$ is too low compared to actual exp. Results.

Actually $\frac{q_u}{T_o} > 8$ because of 3-D problem

3-D Griffith Theory yields:

$$(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2 = 24\ T_o\ (\sigma_1 + \sigma_2 + \sigma_3)$$

For uniaxial unconfined condition, $(\sigma_2 = \sigma_3 = 0)$, we end up with:

$$\sigma_1^2 + \sigma_1^2 = 24\ T_o\ \sigma_1$$

$$= \sigma_1^2 = 24\ T_o\ \sigma_1 \qquad \sigma_1 = 12\ T_o \qquad \text{or} \qquad q_u = 12\ T_o$$

as a rule of thumb: $\frac{q_u}{T_o} \cong 10$

Table 3.1

* Note eqn (5) doesn't account for fraction in case the cracks closed up under hydrostatic pressure.

** Modified Griffith Theory (Mclintock & Walsh, 1962)

They allowed for fraction along the surface of closing cracks.

$$T_o = \tfrac{1}{4}\,\sigma_1\,[(\tan^2\phi + 1)^{1/2} - \tan\phi] - \tfrac{1}{4}\,\sigma_3\,[(\tan^2\phi + 1)^{1/2} + \tan\phi].$$

max. shear stress: $\tau = 2\ T_o + \sigma_N \tan\phi$

Note: this suggests that

$$S_i = 2\ T_o$$

5- Empirical Criteria of Failure: best way for rock.

- do tests
- draw best failure envelope

* draw a failure envelope tangent to experimentally obtained Mohr Circles.

1. Murrel (1965):

$\sigma_1 = q_u + b\,\sigma_3^{m}$

where b & m are constants.

2. Bieniawski (1974) proposed a similar expression:

$$\left(\frac{\sigma_1}{q_u}\right) = 1 + N\left(\frac{\sigma_3}{q_u}\right)^{M}$$

N and M are constants for a given rock, and employ $\sigma_3 = T_o$ tension cut-off.

3. Johnston (1985) proposed:

$$\sigma'_{1n} = \left(\frac{M}{B}\,\sigma'_{3n} + S\right)^{B}$$

where $\sigma_{1n} = \dfrac{\sigma'_1}{q_u}$ $\leftarrow = \sigma_c'$

$$\sigma'_{3n} = \frac{\sigma'_3}{q_u}$$

S = 1.0 for intact rock.

Note: 1. for unconfined compression ($\sigma_3' = 0$) $\Rightarrow$ $\sigma_{1n}' = 1.0$.

2. for uniaxial tensile strength ($\sigma_1' = 0$, $\sigma_3' = \sigma_t'$)

$$\frac{\sigma'_c}{\sigma'_t} = -M/B$$

Note: If B = 1.0 (for over-consolidated clays B = 1.0)

$$\sigma_{1n}' = (M\, \sigma_{3n}' + 1)^{1.0}$$

If $M = \dfrac{1+\sin\phi}{1-\sin\phi}$ we get Mohr-Coulomb theory exactly.

i.e. B = 1.0 $\Rightarrow$ envelope is st. line

if B $\neq$ 1.0 $\Rightarrow$ envelope is not st. line

Paper:

1. How he choose data

2. Criteria for accepting data :
   - uniaxial tensile
   - Brazilian result (Fig. 2.)
   - triaxial test

3. Eliminate some data $\rightarrow$ if splitting instead of shear. (Fig. 1-a)

4. Fig. 3 $\Rightarrow$ B = f (confining pressure)

   $B = 1 - .0172\,(\log \sigma_c')^2$

   where σ_c' in kPa.

   B is independent of rock type.

5. Fig. 4 M is f (confining pressure + rock type)

   a. $M = 2.065 + 0.170\,(\log \sigma_c')^2$: dolomite, limestone, chalk.

   b. $M - 2.065 + 0.231\,(\log \sigma_c')^2$: shale, slate, mudstone, clay.

   c. $M = 2.065 + 0.270\,(\log \sigma_c')^2$: quartzite, sandstone.

   d. $M = 2.065 + 0.659\,(\log \sigma_c')^2$: coarse grained, igneous & metamorphic.

- Fig. 9 $\sigma_c' = f\left(\frac{\sigma_c'}{\sigma_t}\ ,\ \text{rock type}\right)$

- Table 3 $\sigma_c'/\sigma_t = 24.3$ to 2.9

  The softer the rock the lower the σ_c'/σ_t ratio.

$\phi = 20^o - 60^o$ Table 3.3 / pp. 83