Solution of HW # 8

Problem # 1:

Given:

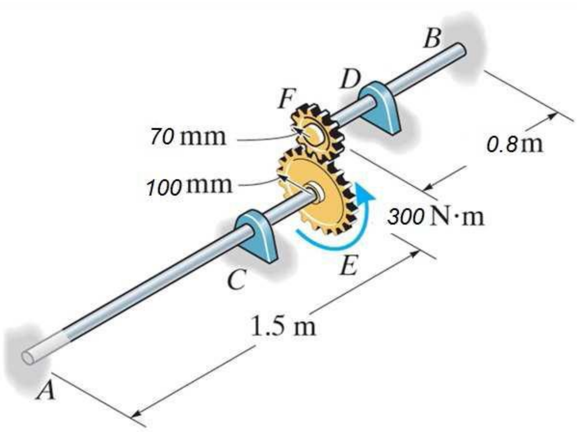
The figure shown

r = 30 mm ; G = 100 GPa

Required:

Value and location of $\boldsymbol{\tau_{max}}$,

$\boldsymbol{\varphi_F}$



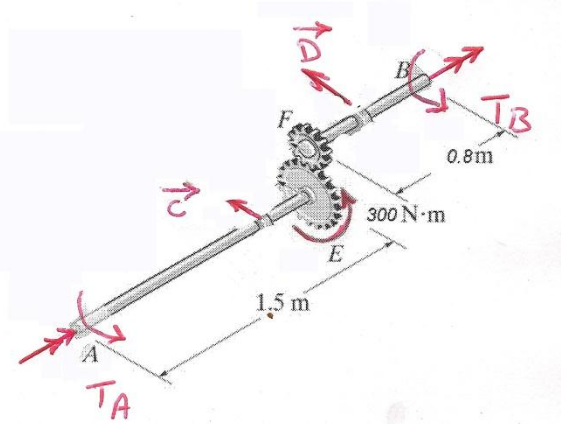
Solution:

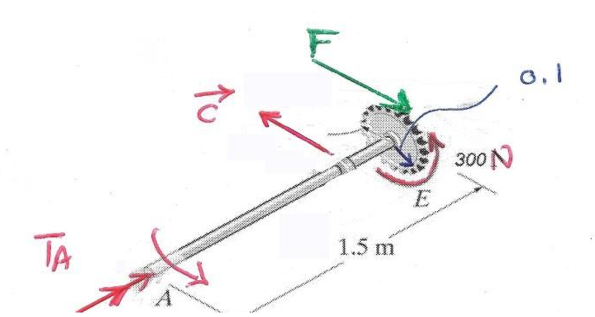
The system is statically indeterminate as it is fixed at A and B; thus, there are two unknowns $\boldsymbol{T_A}$ *and* $\boldsymbol{T_B}$ *(reactions) and only one equilibrium equation (ΣT = 0), as shown below in the FBD.*

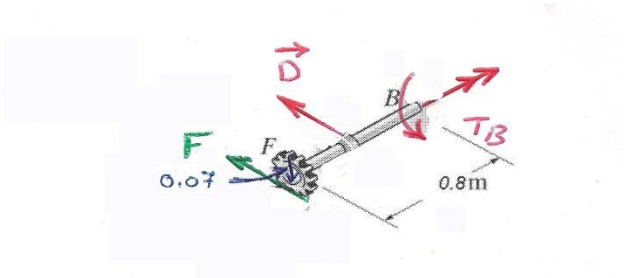
1 *Equilibrium*

$\Sigma\, \boldsymbol{T_{axis}} = 0$

*Since the two shafts are not on the same line, when we take Σ T about the axis of one of them, the reaction on the bearing of the other one will produce **"T"**, and thus the reaction appears on the equation. Therefore, it is better to separate the two shafts from the gear and take each one separately as shown below.*







$$\sum \boldsymbol{T}_{\substack{AE\\axis}} = \boldsymbol{0} \Rightarrow \boldsymbol{T_A - 300 + F(0.1) = 0} \qquad \textbf{(1)}$$

$$\sum \boldsymbol{T}_{\substack{FB\\axis}} = \boldsymbol{0} \Rightarrow \boldsymbol{T_B + F(0.07) = 0} \qquad \textbf{(2)}$$

$From\ (2), \qquad F = -\frac{T_B}{0.07} \Rightarrow$

$T_A - 300 + \left(-\frac{T_B}{0.07}\right)(0.1) = 0 \Rightarrow$

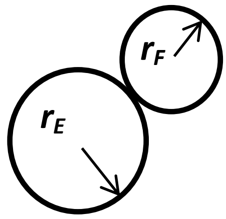
$\boldsymbol{T_A - \frac{10}{7} T_B - 300 = 0}$ **(3)**

(2) *Geometric Compatibility:*

The geometric compatibility of the gears will be used (as discussed in the previous HW).

$r_E \theta_E = r_F \theta_F$

$\theta = \varphi \qquad in\ our\ case\ \Rightarrow$



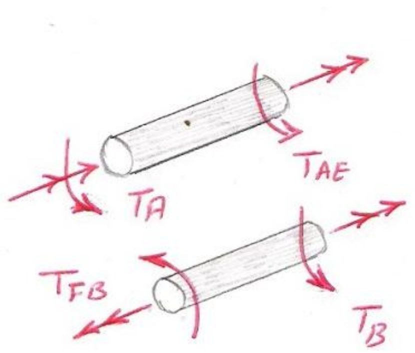
$\boldsymbol{0.1\varphi_E = 0.07\varphi_F \Rightarrow \varphi_E = 0.7\varphi_F}$ **(4)**

$From\ the\ boundary\ conditions,$

$\varphi_E = \varphi_{E/A} = \varphi_{AE} \quad as\ A\ is\ fixed$

$\varphi_F = \varphi_{F/B} = \varphi_{FB} \quad as\ B\ is\ fixed$

$\boldsymbol{Then, eq. (4) becomes} \left(\frac{TL}{JG}\right)_{AE} = \boldsymbol{0.7}\left(\frac{TL}{JG}\right)_{FB}$ **(5)**



$From\ the\ FBD\ shown$

$T_{AE} = -T_A$

$T_{FB} = T_B$

Note that both <u>internal</u> torques are assumed positive. (Why & How?!)

J* & *G *are common in the two shafts, thus eq. (5) becomes* $\quad -1.5T_A = 0.7(0.8T_B) \Rightarrow$

$\boldsymbol{T_B = -2.67857T_B}$ **(6)**

$From\ eq. (6) into\ eq. (3),\ T_A - \frac{10}{7}(-2.67857T_A) - 300 = 0 \Rightarrow$

$\boldsymbol{T_A = 62.156\ N.m} \Rightarrow$

$\boldsymbol{T_B = -166.49\ N.m = 166.49\ N.m}$

Since **r** is the same for the two shafts, $\boldsymbol{\tau_{max}}$ *will be @* $\boldsymbol{T_{max} = T_{FB} = T_B} \Rightarrow$

$$\tau_{max} = \frac{T_{max} r_{max}}{J} = \frac{T_B r_{out}}{J} \Rightarrow$$

$$\tau_{max} = \frac{166.49\ (0.03)}{\frac{\pi}{2}(0.03)^4} \Rightarrow$$

$$\boxed{\boldsymbol{\tau_{max} = 3.926\ MPa\ @\ outer\ raduis\ in\ shaft\ FB}}$$

((Do not worry about the sign!!))

$$\varphi_F = \varphi_{F/B} = \varphi_{FB} \qquad as\ F\ is\ fixed\ \Rightarrow$$

$$\varphi_F = \left(\frac{TL}{JG}\right)_{FB} = \left[\frac{166.49\ (0.8)}{\frac{\pi}{2}(0.03)^4(100)(10)^9}\right] \Rightarrow$$

$\boldsymbol{\varphi_F = 1.0468\ (10)^{-3}\ rad = 0.05998\ ^o}$	**CW**

Problem # 2:

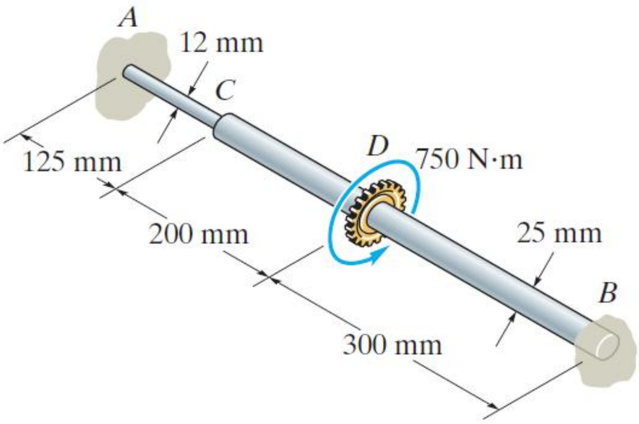
Given:

The figure shown

Required:

$\boldsymbol{T_A}$; $\boldsymbol{T_B}$

$\boldsymbol{\varphi_{D/C}}$; $\boldsymbol{\varphi_{D/B}}$



Solution:

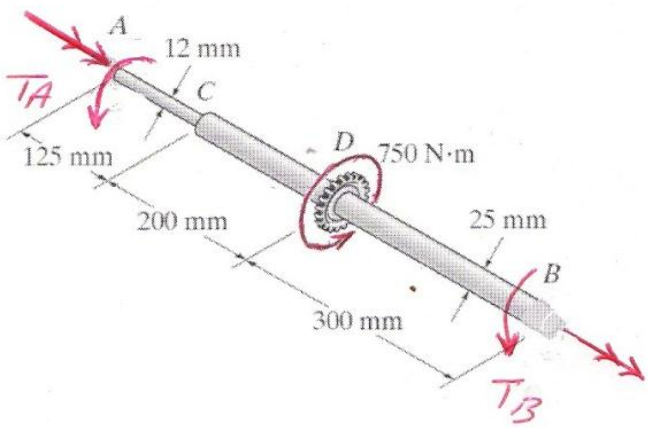
*There are two reactions (at A and B) and only one static (equilibrium) equation (**Σ T = 0**). Thus, the problem is statically indeterminate.*

1 Equilibrium

From the FBD,

$\Sigma\, T_{axis} = 0 \;\Rightarrow$

$\boldsymbol{T_A + T_B + 750 = 0}$ **(1)**



2 Geometric Compatibility:

$\varphi_{A/B} \;\Rightarrow\; \sum \varphi_{A/B} = 0 \;\Rightarrow$

$\varphi_{AC} + \varphi_{CD} + \varphi_{DB} = 0$ (Why 3 segments?!)

$\boldsymbol{\left(\frac{TL}{JG}\right)_{AC} + \left(\frac{TL}{JG}\right)_{CD} + \left(\frac{TL}{JG}\right)_{DB} = 0}$ **(2)**

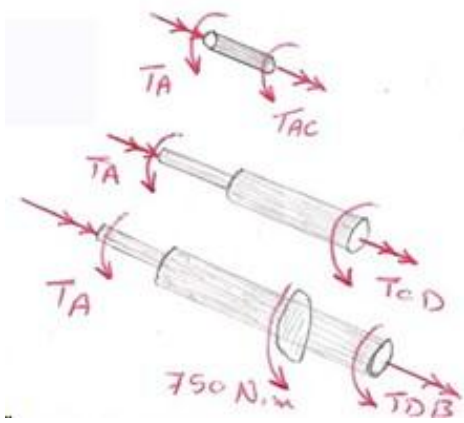
*Since all segments are made of the same material (Steel with **G = 75 GPa**), G can be dropped from the equation.*

***T's** can be obtained from the FBD's below.*

$T_{AC} = -T_A$

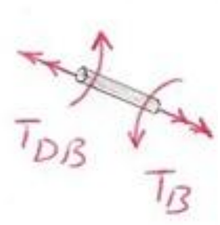
$T_{CD} = -T_A$

$T_{DB} = -T_A - 750$



Note that all internal torques are assumed **(+)**. *Also note that the right part of segment DB may be chosen as shown.*

In this case $T_{DB} = + T_B$.



We can check this later.

Now, eq. (2) can be written as

$$\frac{0.125\,(-T_A)}{\frac{\pi}{2}\left(\frac{0.012}{2}\right)^4} + \frac{0.2\,(-T_A)}{\frac{\pi}{2}\left(\frac{0.025}{2}\right)^4} + \frac{0.3\,(-T_A - 750)}{\frac{\pi}{2}\left(\frac{0.025}{2}\right)^4} = 0 \quad \Rightarrow$$

$$\boxed{\boldsymbol{T_A = -78.816\ N.m = 78.816\ N.m} \qquad \text{((as expected!))}} \quad \Rightarrow$$

$$\boxed{\boldsymbol{T_B = -671.18 = 671.18\ N.m} \qquad \text{((as expected!))}}$$

Note that $\boldsymbol{T_B}$ ***is noticeably bigger than*** $\boldsymbol{T_A}$ ***. Reasonable?! Why?!***

$$\varphi_{D/C} = \left(\frac{TL}{JG}\right)_{CD} = \frac{78.816\ (0.2)}{\frac{\pi}{2}\left(\frac{0.025}{2}\right)^4 (75)(10)^9} \Rightarrow$$

$$\boxed{\boldsymbol{\varphi_{D/C} = 5.4805\ (10)^{-3}\ rad = 0.3140^o}}$$

$$\varphi_{D/B} = \left(\frac{TL}{JG}\right)_{DB}$$

$$T_{DB} = -T_A - 750 = -(-78.816) - 750 = -671.18\ N.m = T_B$$ <u>***Ok!***</u>

$$\varphi_{D/B} = \frac{671.18\ (0.3)}{\frac{\pi}{2}\left(\frac{0.025}{2}\right)^4 (75)(10)^9} \Rightarrow$$

$$\boxed{\boldsymbol{\varphi_{D/B} = 0.070\ rad = 4.011^o}}$$

<u>***Check:***</u>

When we add $\boldsymbol{\varphi_{C/A}}$ *to* $\boldsymbol{\varphi_{D/C}}$ *we should end up with* $\boldsymbol{\varphi_{D/A}}$ *which should equal to* $\boldsymbol{\varphi_{D/B}}$. $\Rightarrow$

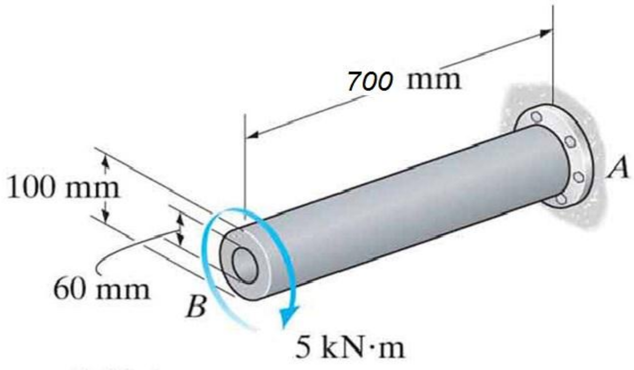
$$\varphi_{C/A} = \left(\frac{TL}{JG}\right)_{AC} = \frac{-(-78.816)(0.125)}{\frac{\pi}{2}\left(\frac{0.012}{2}\right)^4 (75)(10)^9} = 0.064527\ rad = 3.6971^o$$

$$\varphi_{D/A} = \varphi_{C/A} + \varphi_{D/C} = 3.6971 + 0.3140 = 4.011^o = \varphi_{D/B} \Rightarrow$$ <u>***Ok!***</u>

Problem # 3:

Given:

The figure shown



Core: solid steel; d = 60 mm; G = 80 GPa

Tube: brass; G = 40 GPa

Required:

Value and location of $\boldsymbol{\tau_{max\,\&\,min}}$ *in steel and brass*

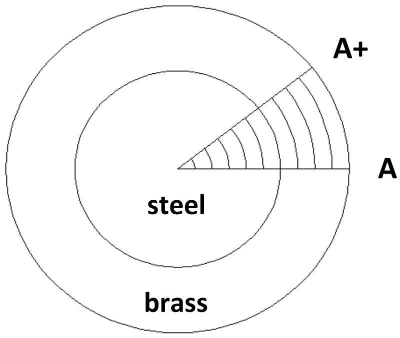
$\boldsymbol{\tau}$ *distribution*

Solution:

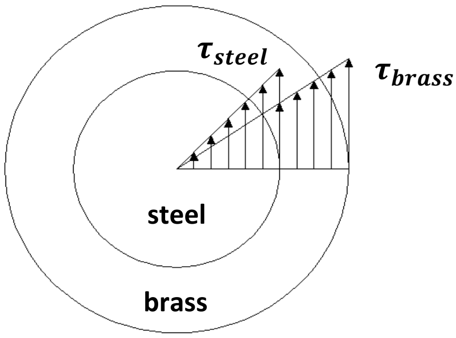
The problem is internally statically indeterminate; that is the 5 kN.m – T is carried by steel and brass, but we do not know $\boldsymbol{T_{steel}}$ *&* $\boldsymbol{T_{brass}}$*.*

We know that the shear strain (γ) must be continuous through the radius as shown below.

For the shear stress ($\boldsymbol{\tau}$*), it is equal to* **G** *γ. Since* **G** *changes on the border of steel/brass,* $\boldsymbol{\tau}$ *will be discontinuous at that point.*

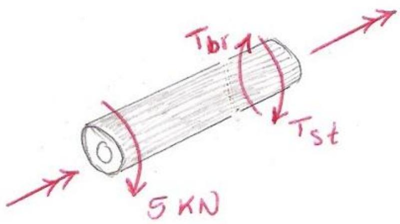


$\boldsymbol{\gamma - distribution}$



$\boldsymbol{\tau - distribution}$

From the FBD (equilibrium)



$\boldsymbol{T_{st} + T_{br} + 5000 = 0}$ **(1)**

From the geometric compatibility,

$\boldsymbol{\varphi_{st} = \varphi_{br}}$ **(2)**

((as steel and brass are bounded together, they must have the same rotation.))

From material behavior,

$\boldsymbol{\varphi = \frac{TL}{JG}}$ **(3)**

$From\ (3) into\ (2),\qquad \left(\frac{TL}{JG}\right)_{st} = \left(\frac{TL}{JG}\right)_{br} \Rightarrow$

$$\frac{T_{st}L}{\frac{\pi}{2}\left(\frac{60}{2}\right)^4 (80)(10)^9} = \frac{T_{br}L}{\frac{\pi}{2}\left[\left(\frac{100}{2}\right)^4 - \left(\frac{60}{2}\right)^4\right](40)(10)^9} \Rightarrow$$

$\boldsymbol{T_{br} = 3.35802 T_{st}}$ **(4)**

From (4) into (1), $\Rightarrow \boldsymbol{T_{st} = 1147.31\ N.m}$

$\Rightarrow$ *into (4)* $\Rightarrow \boldsymbol{T_{br} = 3852.69\ N.m}$

Note that to take advantages of both materials, it is better to have the brass as the core (inner) material and the steel as the sleeve/tube (outer) material. (Why & How?!)

$$\tau_{max} = \frac{T_{max} r_{max}}{J}$$

Since we have only one external $\boldsymbol{T}$, $\boldsymbol{T}$ *will be constant, and* $\boldsymbol{\tau_{max}}$ *@* $\boldsymbol{r_{max}}$ *and* $\boldsymbol{\tau_{min}}$ *@* $\boldsymbol{r_{min}}$.

Thus,

$$\boldsymbol{\tau_{max}^{st} = \frac{T_{st} r_{max}}{J_{st}} = \frac{1147.31\ (0.03)}{\frac{\pi}{2}(0.03)^4}}$$

$$\boxed{\boldsymbol{\tau_{max}^{st} = 27.05\ MPa}}$$

$$\tau_{min}^{st} = \frac{T_{st} r_{min}}{J_{st}} = \frac{T\ (0)}{J}$$

$$\boxed{\boldsymbol{\tau_{min}^{st} = 0}}$$

$$\boldsymbol{\tau_{max}^{br} = \frac{3852.69\ (0.05)}{\frac{\pi}{2}[(0.05)^4 - (0.03)^4]}} \Rightarrow$$

$$\boxed{\boldsymbol{\tau_{max}^{br} = 22.54\ MPa}}$$

$$\tau_{min}^{br} = \frac{3852.69\,(0.03)}{\frac{\pi}{2}[(0.05)^4 - (0.03)^4]} \Rightarrow$$

$$\boxed{\tau_{min}^{br} = 13.53\ MPa}$$

Check:

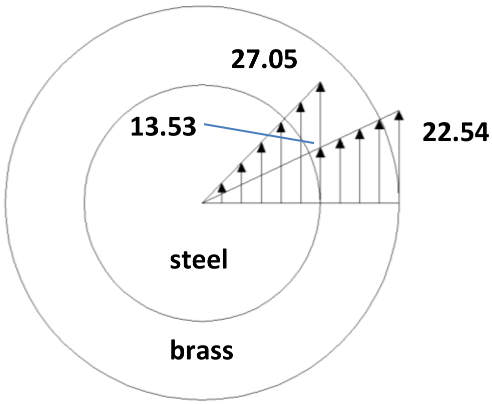
$\gamma_{st} = \gamma_{br}$ @ $r = 0.03 \Rightarrow \left(\frac{\tau}{G}\right)_{st} = \left(\frac{\tau}{G}\right)_{br} \Rightarrow$

$$\frac{\tau_{st}}{\tau_{br}} = \frac{G_{st}}{G_{br}} \Rightarrow$$

$$\frac{\tau_{st}}{\tau_{br}} = \frac{27.05}{13.53} = 2.00$$

$$\frac{G_{st}}{G_{br}} = \frac{80}{40} = 2.00$$

OK



$\tau - distribution\ in\ MPa$

Problem # 4:

Given:

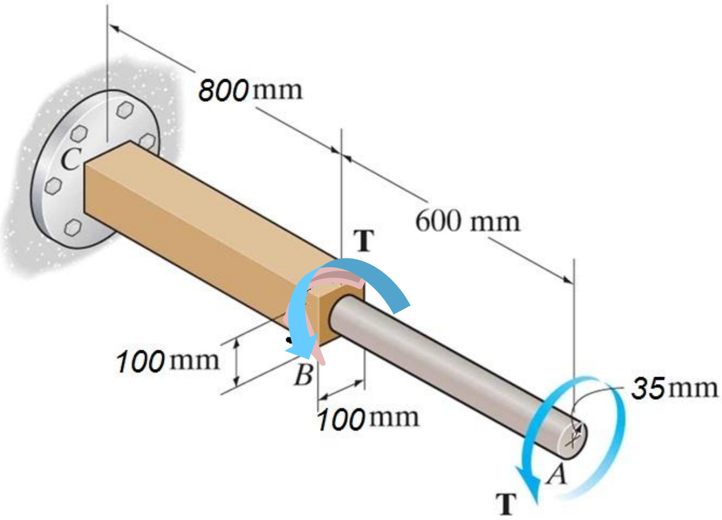
The figure shown

$\tau_{allow} = 120\ MPa; \quad \varphi^{A}_{max} = 1^{o}$;

$\varphi^{B}_{max} = 0.5^{o}; \quad G = 100\ GPa$

Required:

Maximum allowable $\boldsymbol{T}$



Solution:

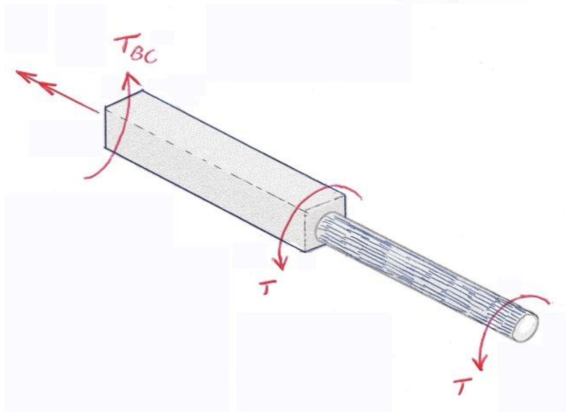
Here, we have four criteria (how?!) we need to satisfy. We calculate $\boldsymbol{T}_{\substack{max\\allow}}$ *for each, and then we choose the smallest value for the answer* $\boldsymbol{T}_{\substack{max\\allow}}$. *(Why?! see previous HW!).*

The problem is statically determinate so that we can find the *internal* $\boldsymbol{T}$ *directly. We have* *two segments* *AB and BC. (Why?!)*

From the FBD's,

$T_{AB} = T$

$T_{BC} = 2T$



For segment AB, set $\tau_{max} \equiv 120\ MPa$.

$$\tau_{max} = \frac{T_{AB} r_{max}}{J_{AB}} = \frac{T_{max}\ (0.035)}{\frac{\pi}{2}\ (0.035)^4} \equiv 120\ (10)^6$$

$$\Rightarrow\ T^{1}_{max} = 8.082\ kN.m$$

For segment BC, set $\tau_{max} \equiv 120\ MPa$.

$$\tau_{max} = \frac{4.81\ T_{BC}}{a^3} = \frac{4.81\ (2T_{max})}{(0.1)^3} \equiv 120\ (10)^6$$

$$\Rightarrow\ T^{2}_{max} = 12.47\ kN.m$$

Now consider φ^{B}, *then* φ^{A}. *(Why?!)*

$$\varphi^B = \frac{7.10\, T_{BC} L_{BC}}{a^4 G} = \frac{7.10\,(2T_{max})(0.8)}{(0.1)^4(100)(10)^9} \equiv 0.5\left(\frac{\pi}{180}\right)$$

$$\Rightarrow\ T^3_{max} = 7.682\ kN.m$$

$$\varphi^A = \sum \varphi = \left(\frac{TL}{JG}\right)_{AB} + \left(\frac{7.10\ TL}{aG}\right)_{BC} = \frac{T_{max}(0.6)}{\frac{\pi}{2}(0.035)^4(100)(10)^9} + \frac{7.10\,(2T_{max})(0.8)}{(0.1)^4(100)(10)^9}$$

$$2.54542\,(10)^{-6}\, T_{max} + 1.136\,(10)^{-6}\, T_{max} \equiv 1\left(\frac{\pi}{180}\right)$$

$$\Rightarrow\ T^4_{max} = 4.741\ kN.m$$

T^4_{max} *due to* φ^A_{max} *controls.* $(Why?!)$

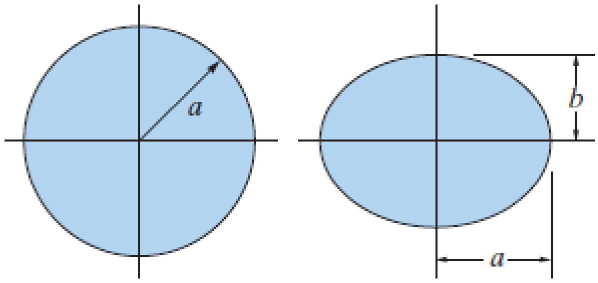
$$\Rightarrow \quad \boxed{\boldsymbol{T_{\substack{max \\ allow}} = 4.741\ kN.m}}$$

Problem # 5:

Given:

The figure shown

a = 30 mm; b = 15 mm; T = 80 N.m



Required:

$\boldsymbol{\tau_{max}}$ *in the two sections*

Efficiency of circular section compared with the elliptical one.

Solution:

$$\tau_{max}^{cir.} = \frac{T r_{max}}{J} = \frac{80\ (0.03)}{\frac{\pi}{2}(0.03)^4} \Rightarrow \qquad \boxed{\boldsymbol{\tau_{max}^{cir.} = 1.886\ MPa}}$$

$$\tau_{max}^{ell.} = \frac{2\ T}{\pi a b^2} = \frac{2\ (80)}{\pi(0.03)(0.015)^2} \Rightarrow \qquad \boxed{\boldsymbol{\tau_{max}^{ell.} = 7.545\ MPa}}$$

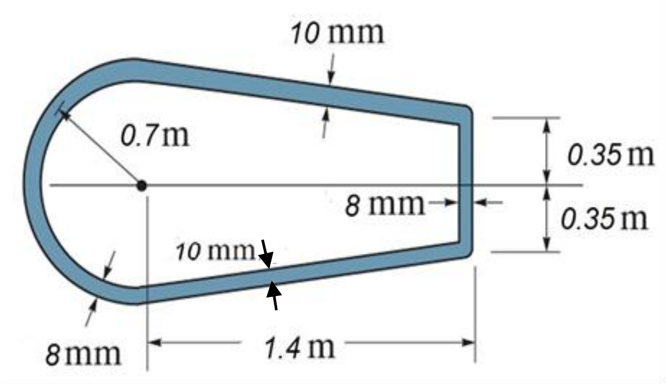
% max efficiency (cir./ell.) = $\frac{\tau_{max}^{ell.} - \tau_{max}^{cir.}}{\tau_{max}^{cir.}}\ X\ 100$

$$= \frac{7.545 - 1.886}{1.886} X\ 100$$

$\boxed{\textit{Efficiency = 300 \%}}$

Problem # 6:

Given:



The cross-section of a shaft shown

T = 300 N.m;   G = 100 GPa

Required:

Value and location of $\boldsymbol{\tau_{max}}$

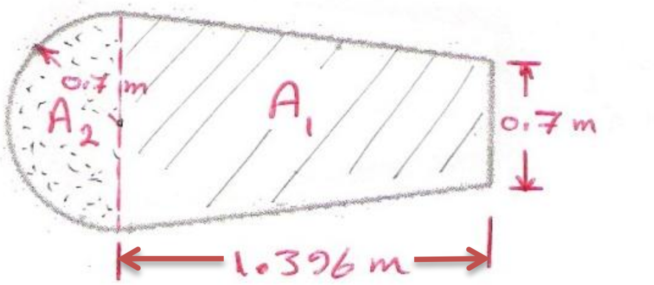
$d\varphi / dz$

Solution:

The section is "thin-walled closed". $\Rightarrow$

$$\tau = \frac{T}{2tA_m}$$

$\boldsymbol{A_m}$ *is the area contained within the* *mean perimeter* (*not material area*) *as shown in the below figure.*



$$A_m = A_1 + A_2$$

$$= \frac{0.7 + 2\,(0.7)}{2}(1.396) + \frac{\pi}{2}\,(0.7)^2$$

$$= 2.23549\; m^2$$

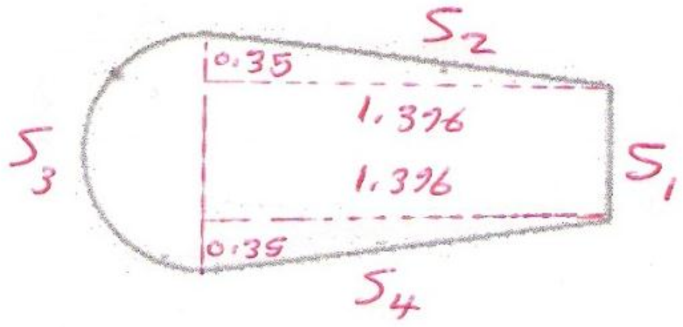
Since t is in the denominator of the $\boldsymbol{\tau}$ *formula,* $\boldsymbol{\tau_{max}}$ *will be at* t_{min} *= 8 mm* $\Rightarrow$

$$\tau_{max} = \frac{300\;(10)^3}{2\;(0.008)(2.23549)} \Rightarrow$$

$$\boxed{\boldsymbol{\tau_{max} = 8.387\; MPa\; @\; the\; 8 - mm\; thicknes}}$$

$$\frac{d\varphi}{dz} = \frac{T}{4A_m^2 G}\oint \frac{ds}{t}$$

$$= \frac{T}{4A_m^2 G}\sum_{i=1}^{4}\frac{s_i}{t_i} \qquad \textit{in our case (Why?!)}$$



$S_1 = 0.7\,m \quad ; \quad t_1 = 0.008\,m$

$S_2 = \sqrt{(1.396)^2 + (0.35)^2}$

$= 1.43921\,m = S_4$

$t_2 = t_4 = 0.01\,m$

$S_3 = \pi r = \pi(0.7) = 2.19911\,m \quad ; \quad t_3 = 0.008\,m$

$$\frac{d\varphi}{dz} = \frac{300\,(10)^3}{4\,(2.23549)^2(100)(10)^9}\left(\frac{0.7}{0.008} + 2\frac{1.43921}{0.01} + \frac{2.19911}{0.008}\right)$$

$$\boxed{\boldsymbol{\frac{d\varphi}{dz} = 9.759\,(10)^{-5}\ {}^{rad}/_{m} = 5.591\,(10)^{-3}\ {}^{o}/_{m}}}$$