

Transformation of Stress

Theory & Examples

* **Triaxial** states of stress are shown in Fig. (1).

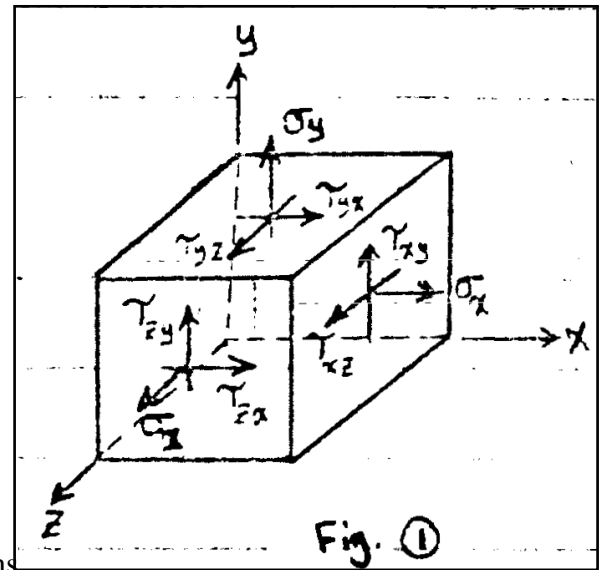
Only positive stresses on positive faces are shown

* **Biaxial** states of stress (Plane Stress):

When all stresses act in the same plane.

⇒ Work in 2-D as shown in Fig. (2) in the

x-y directions.

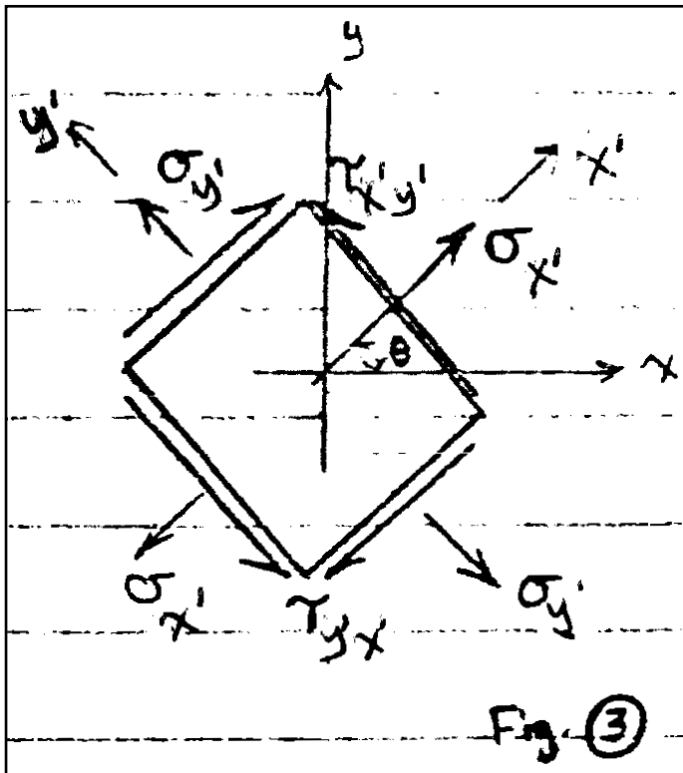
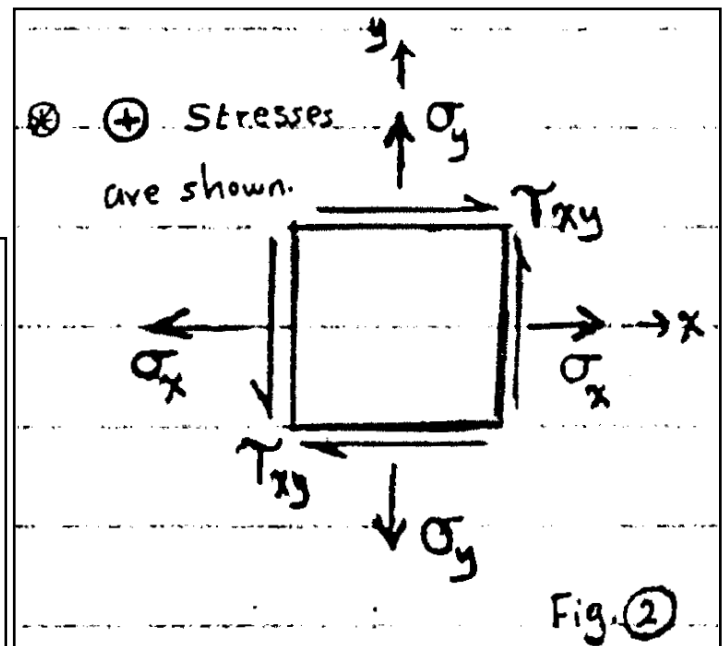


It is possible to have the orientation in different directions,

as shown in Fig. (3) in the x'-y' directions.

Now, relationships between the stresses in the

x-y and x'-y' directions are sought.



From Fig. (4), the sum of **forces** in the x' and y' directions must be zero.

* **Note that forces not stresses are added.**

(2 eqs. & 2 unkns.)

$$\sum F_{x'} = 0 \quad \& \quad \sum F_{y'} = 0 \quad \Rightarrow$$

$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

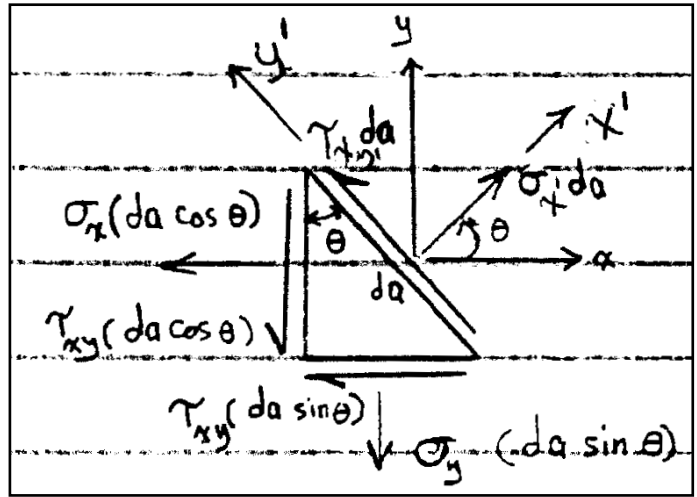


Fig. (4)

Recall that

$$\cos^2 \theta = (1 + \cos 2\theta)/2$$

$$\sin^2 \theta = (1 - \cos 2\theta)/2$$

$$\sin \theta \cos \theta = \sin 2\theta/2$$

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta$$

\Rightarrow

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad [1]$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad [2]$$

Similarly,

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad [3]$$

$$\tau_{y'x'} = \tau_{x'y'} \quad [4]$$

* **Note that $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'} = I$ (Invariant with any θ)**

\Leftarrow Use it to check.

When σ & τ on any two orthogonal faces are known, the stress components on all (any) faces (plane stress) can be calculated.

Principal Normal Stresses:

$\sigma_{x'} = f(\theta) \Rightarrow$ to get $\sigma_{x'_{max}}$, set $\frac{d\sigma_{x'}}{d\theta} = 0 \Rightarrow$ find $\theta_p \Rightarrow \sigma_{x'_{max}}$

$$\frac{d\sigma_{x'}}{d\theta} = -(\sigma_x - \sigma_y) \sin 2\theta_p + 2\tau_{xy} \cos 2\theta_p = 0$$

Dividing by $\cos 2\theta$,
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \quad [5]$$

From the equation above, Fig. (5) shown can be constructed.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad [6]$$

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}$$

$$\cos 2\theta_{p1} = \frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\sin 2\theta_{p2} = \frac{-\tau_{xy}}{R} \quad ; \quad \cos 2\theta_{p2} = \frac{-(\sigma_x - \sigma_y)/2}{R}$$

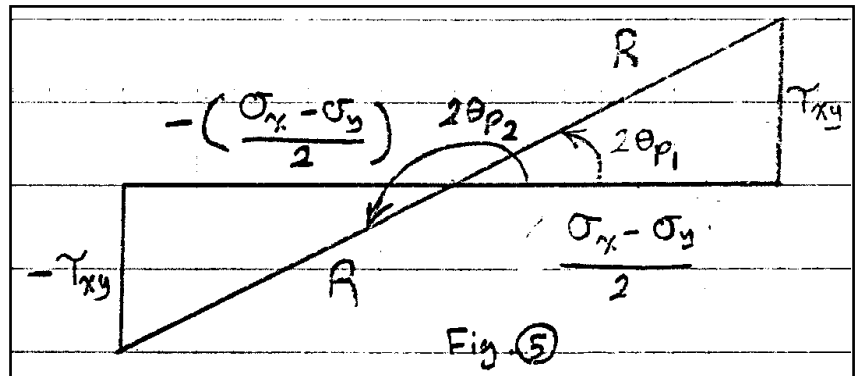
$$\Rightarrow \sigma_{\max/\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_x + \sigma_y}{2} \pm R \quad [7]$$

↑ Principal Normal Stresses

The directions are given by θ_{p1} and θ_{p2}

Note that $2\theta_{p2} = 2\theta_{p1} + \pi \Rightarrow \theta_{p1} \perp \theta_{p2}$

Also note that $\tau_{x'y'} = 0$ on the planes which the principal normal stresses act.



maximum Shear Stresses: \Leftarrow sometimes called **principal τ**

$\tau_{x'y'} = f(\theta) \Rightarrow$ The value of θ_s can be obtained by setting $\frac{d\tau_{x'y'}}{d\theta} = 0$.

$$\Rightarrow \tan 2\theta_s = \frac{-\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}} \quad [8]$$

[Fig. (6)]

There are 2 possible values for θ_s

$$\Rightarrow \sin 2\theta_{s1} = \frac{(\sigma_x - \sigma_y) / 2}{R}$$

$$\cos 2\theta_{s1} = \frac{-\tau_{xy}}{R}$$

$$\sin 2\theta_{s2} = \frac{-(\sigma_x - \sigma_y) / 2}{R} \quad ; \quad \cos 2\theta_{s2} = \frac{\tau_{xy}}{R}$$

$$\Rightarrow \tau_{\max} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm R \quad [9]$$

\uparrow Maximum (Principal) Shear Stresses

$$\sigma_{x'} = \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} \quad [10]$$

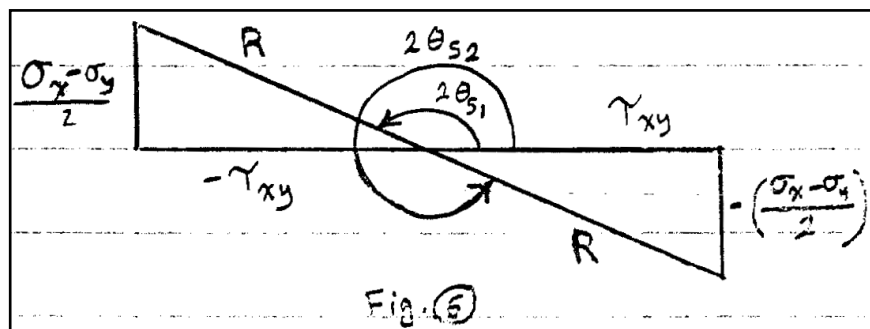
\uparrow Normal Stresses on the Planes of Maximum (Principal) Shear Stresses

Note that $2\theta_{s2} = 2\theta_{s1} + \pi \Rightarrow \theta_{s1} \perp \theta_{s2}$

Also note that $\tan 2\theta_p$ is the negative reciprocal of $\tan 2\theta_s$: $\tan 2\theta_p = -1/\tan 2\theta_s$

$$\Rightarrow 2\theta_s = 2\theta_p + \pi/2 \Rightarrow \theta_s = \theta_p + 45^\circ \quad [11]$$

Thus, there is a 45° -angle between the planes of principal normal and maximum shear stresses.



Example 1:

Given:

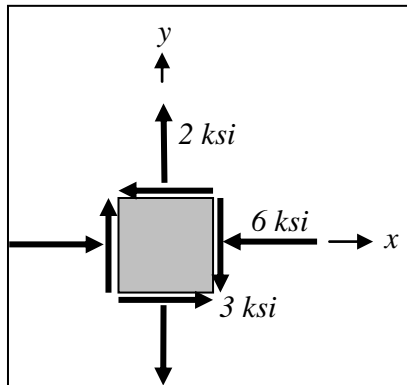
The state of stress shown

Req'd:

- a) The principal stresses & directions
- b) σ & τ associated with an element oriented 10° cw of the element shown.

Show the results on properly oriented elements.

Use the equations for the solution.

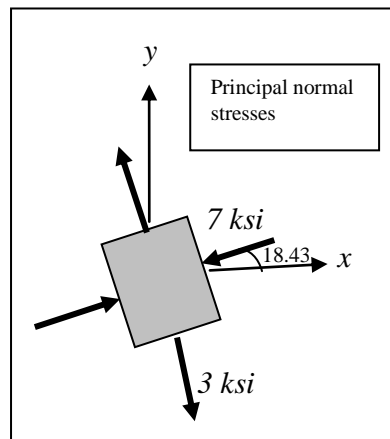


Solution:

a)

$$\sigma_{\max/\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-6 + 2}{2} \pm \sqrt{\left(\frac{-6 - 2}{2}\right)^2 + (-3)^2} = -2 \pm 5$$



$\Rightarrow \sigma_{\max} = 3 \text{ ksi} \quad ; \quad \sigma_{\min} = -7 \text{ ksi}$

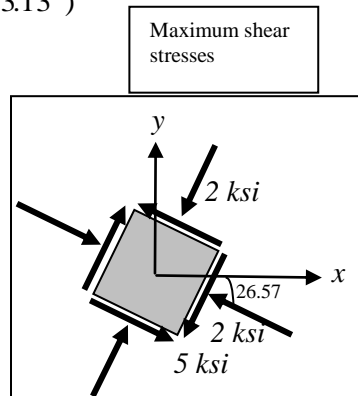
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-3)}{-6 - 2} = 0.75 \Rightarrow 2\theta_{p1} = 36.87^\circ, \quad 2\theta_{p2} = 216.87^\circ \quad (-143.13^\circ)$$

$\Rightarrow \theta_{p1} = 18.43^\circ \quad ; \quad \theta_{p2} = 108.43^\circ \quad (-71.57^\circ)$

To see θ_{p1} corresponds to σ_{\max} or σ_{\min} , substitute θ in Eq. [1] by $\theta_{p1} = 18.4349^\circ$

$$\Rightarrow \sigma_{x'} = \frac{-6 + 2}{2} + \frac{-6 - 2}{2} \cos 36.87^\circ - 3 \sin 36.87^\circ = -7 \text{ ksi}$$

$\Rightarrow \theta_{p1}$ is the direction of σ_{\min} as shown.



$$\tau_{\max/\min} = \pm R \Rightarrow \tau_{\max} = 5 \text{ ksi} \quad ; \quad \tau_{\min} = -5 \text{ ksi}$$

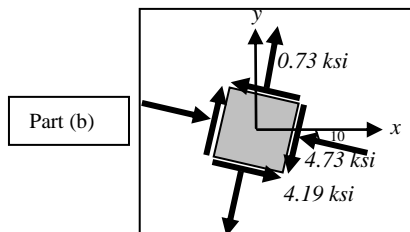
$$\sigma_{x'} = \sigma_{y'} = \frac{-6 + 2}{2} = -2 \text{ ksi}$$

$$\tan 2\theta_s = -1/0.75 \Rightarrow \theta_{s1} = -26.57^\circ \quad ; \quad \theta_{s2} = 63.43^\circ \quad ; \quad \tau(-26.57^\circ) = -5 \text{ ksi} = \tau_{\min} \leftrightarrow \theta_{s1}$$

b) From. Eqs. [1] to [4], $\sigma_{x'} (-10^\circ) = -4.73 \text{ ksi} \quad ; \quad \sigma_{y'} (-10^\circ) = 0.73 \text{ ksi}$

Note that $\sum \sigma_i = -4$ (always)

$\tau_{x'y'} (-10^\circ) = \tau_{y'x'} (-10^\circ) = -4.19 \text{ ksi} (\pm)$



Mohr's Circle:

The general equation of the circle is

$$(x-a)^2 + (y-b)^2 = R^2 \quad [12]$$

By rearranging equation [1] and squaring both sides of equations [1] & [2], and then adding, the following equation is obtained:

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} \right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \quad [13]$$

By comparing eq. [13] with eq. [12], it can be seen that

$$x = \sigma_{x'}$$

$$a = (\sigma_x + \sigma_y)/2 = \sigma_{average}$$

$$y = \tau_{x'y'}$$

$$b = 0$$

$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2$$

Thus, by constructing a circle with the properties above, and by referring to Fig. (5), the values of $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ with any θ can be found; this includes the principal stresses and their directions.

This circle is called Mohr's Circle because Mohr brought the idea of such a circle. It has several applications, other than stresses.

Steps for constructing Mohr's circle and determining the principal stresses & directions and σ & τ with any θ :

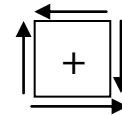
- (1) Draw the σ - τ axes in the x - y (i.e., horizontal-vertical) directions with "appropriate" scale. Note the direction of $+\tau$ (down).
- (2) Put the points x (σ_x, τ_{xy}) and y ($\sigma_y, -\tau_{xy}$) on the figure.
- (3) Connect the two points x and y by a straight line. The point of intersection of the line xy and the σ -axis is the center of the circle C , and Cx & Cy are two radii of such a circle.
- (4) Construct the circle with C as its center and Cx (or Cy) the radius.
- (5) The points of intersection of the circle and the σ -axis are the principal normal stresses; the one to the right is the maximum, and the one to the left is the minimum.
- (6) The radius of the circle is τ_{max} , and $\tau_{min} = -R$.
- (7) The angle measured from the x -axis to σ_{max} gives $2\theta_{p1}$ (or $p2$) (i.e., double the angle), and the angle measured from x to σ_{min} is $2\theta_{p2}$ (or $p1$).
- (8) The angle measured from x to τ_{max} is $2\theta_{s1}$ (or $s2$) and the angle from x to τ_{min} is $2\theta_{s2}$ (or $s1$).
- (9) The similar triangles show in Fig. (7) are used to calculate the required values (stresses and their directions).
- (10) To calculate the stresses on planes oriented θ° from the x -axis on the real plane, go 2θ from the x -axis on the imaginary plane (Mohr's circle), and then draw a straight line passing through C . Use the triangles shown in Fig. (7) to determine the stress values.

Remember:

* **Start from x, Double the angle**

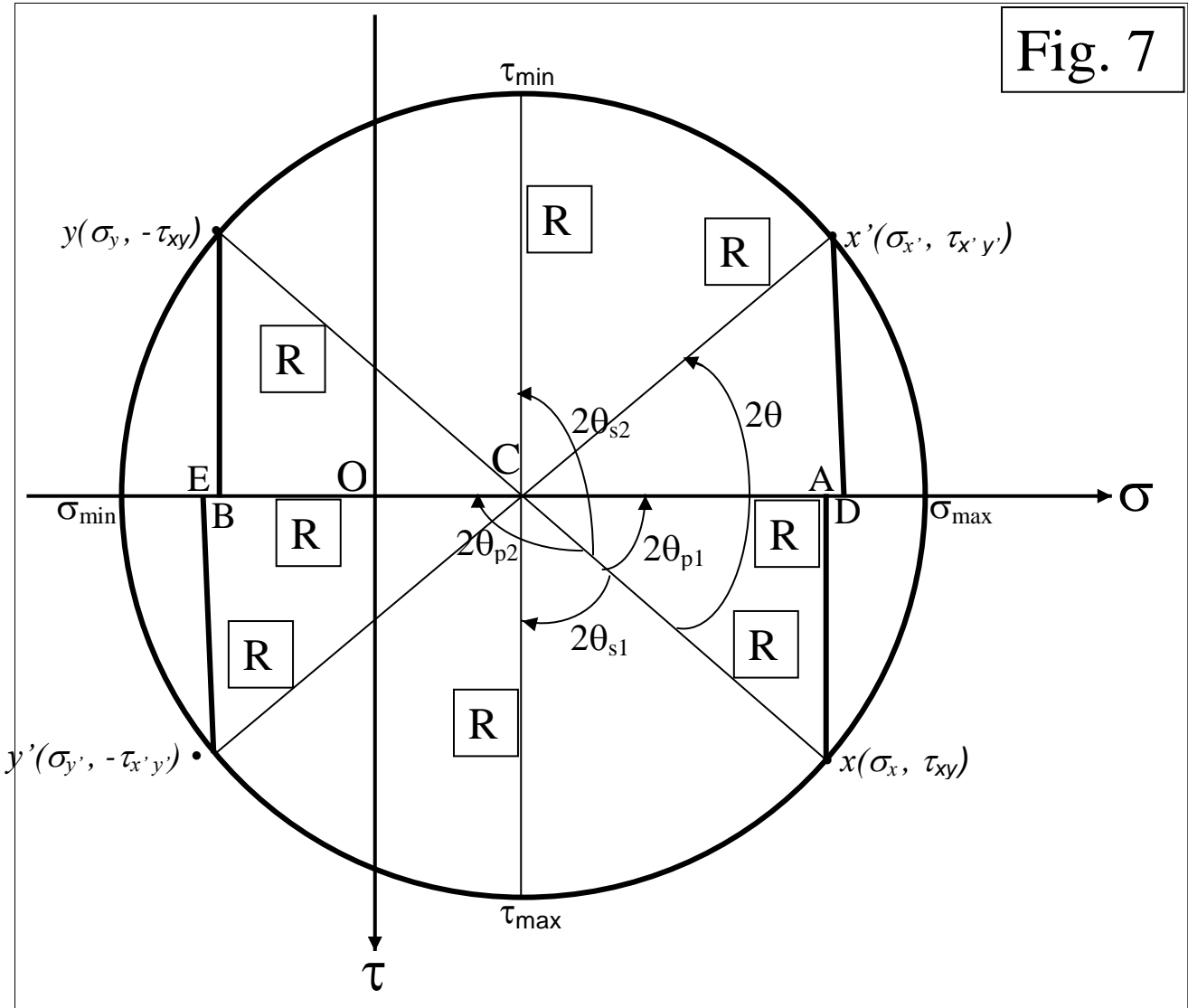
- * Some books use $+\tau$ as up (not down as done here). If it is so, then all angles will be in the opposite directions. You can go in the same direction (not the opposite) if one of the following is done:

(1) Reverse the "sign convention" for shear. \Rightarrow



(2) Plot $x(\sigma_x, -\tau_{xy})$, $y(\sigma_y, \tau_{xy})$.

(3) Plot $+\tau$ axis down (as done here).

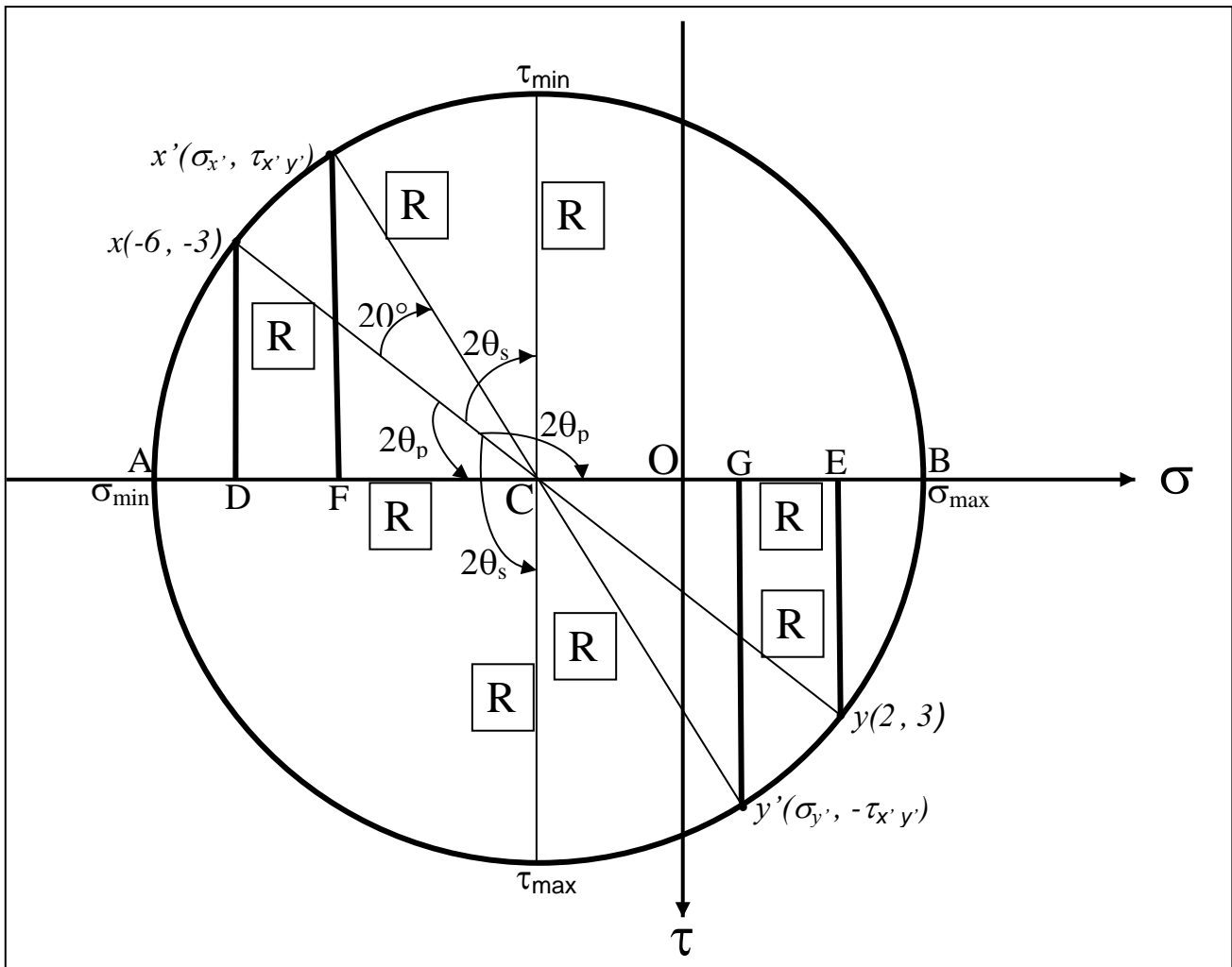
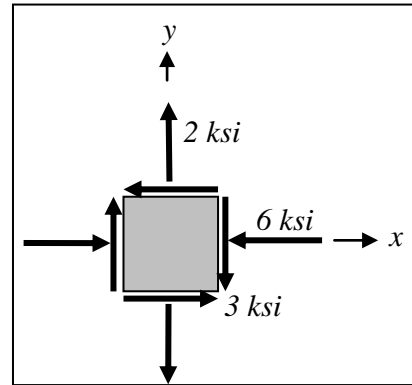


Example 2:

Rework Example 1 using Mohr's circle.

Solution:

a) The steps given above will be followed.



$$OC = \frac{OE + OD}{2} = \frac{2 - 6}{2} = -2$$

Using the triangle CxD or CyE, R and $2\theta_p$ can be calculated. \Rightarrow

$$\tan 2\theta_{p1} = \frac{x_D}{CD} = \frac{|x_D|}{|OD| - |OC|}$$

$$= \frac{3}{6 - 2} = 0.75$$

$$\Rightarrow 2\theta_{p1} = 36.87^\circ \Rightarrow$$

$$\Rightarrow \theta_{p1} = 18.43^\circ \text{ ccw } (\curvearrowleft) \text{ measured from the x-axis to the axis of } \sigma_{\min}$$

* Remember: Double the angle measured from x

$$\Rightarrow 2\theta_{p2} = 180^\circ - 2\theta_{p1}$$

$$\Rightarrow \theta_{p2} = 71.57^\circ \text{ CW } (\curvearrowright) \text{ measured from x to } \sigma_{\max}$$

$$R = C_x = C_y = |x_D| / \sin 2\theta_{p1} = 3 / \sin 36.87^\circ = 5$$

$$\sigma_{\max} = OB = CB - |OC| = R - |OC| = 5 - 2 \Rightarrow \underline{\sigma_{\max} = 3 \text{ ksi}}$$

$$\sigma_{\min} = -|OA| = -(|OC| + |CA|) = -(|OC| + R) = -(2 + 5) \Rightarrow \underline{\sigma_{\min} = -7 \text{ ksi}}$$

Take care of the signs by inspection.

$$\tau_{\max} = \pm R \Rightarrow \underline{\tau_{\max} = 5 \text{ ksi}} \quad ; \quad \underline{\tau_{\min} = -5 \text{ ksi}}$$

$$2\theta_{s1} = 90^\circ - 2\theta_{p1} = 90 - 36.87^\circ \Rightarrow \underline{\theta_{s1} = 26.57^\circ \text{ cw } (\curvearrowright) \text{ measured from x to } \tau_{\min}}$$

$$2\theta_{s2} = 90^\circ + 2\theta_{p1} = 90 + 36.87^\circ \Rightarrow \underline{\theta_{s2} = 63.43^\circ \text{ ccw } (\curvearrowleft) \text{ measured from x to } \tau_{\max}}$$

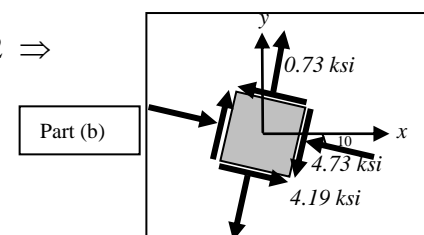
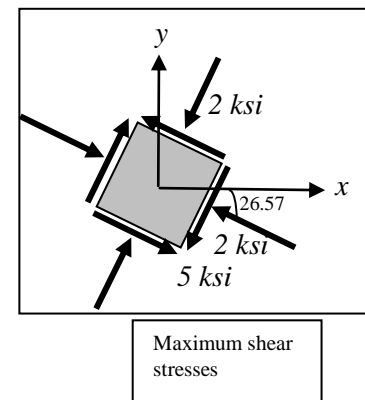
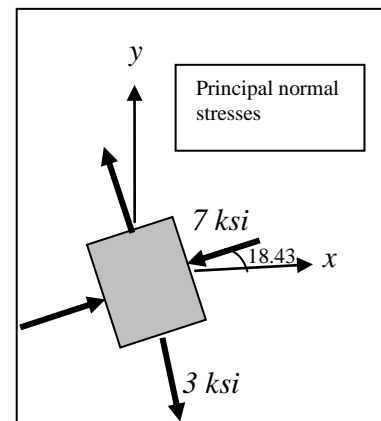
$$b) \sigma_{x'}(-10^\circ) = OF = -(|OC| + |CF|) = -[|OC| + R \cos(2\theta_{p1} + 20^\circ)] = -[2 + 5 \cos(56.87^\circ)] \Rightarrow$$

$$\underline{\sigma_{x'}(-10^\circ) = -4.73 \text{ ksi}}$$

$$\sigma_{y'}(-10^\circ) = OG = R \cos(2\theta_{p1} + 20^\circ) - |OC| = 5 \cos(56.87^\circ) - 2 \Rightarrow$$

$$\underline{\sigma_{y'}(-10^\circ) = 0.73 \text{ ksi}}$$

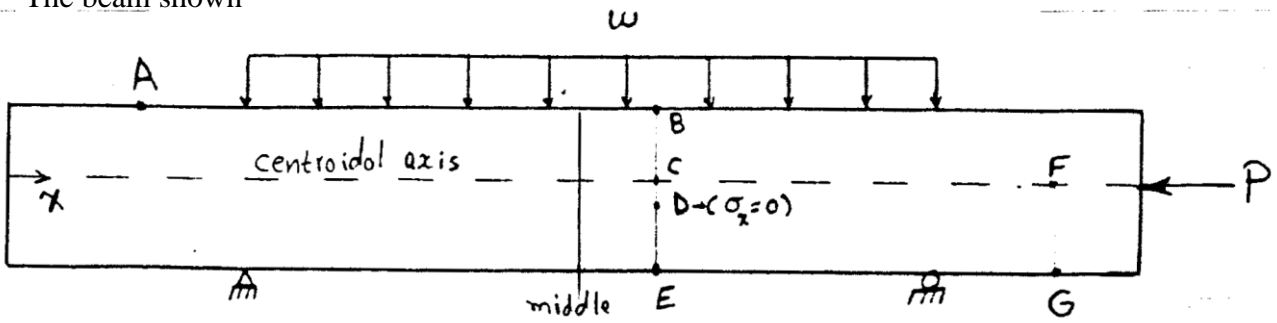
$$\tau_{x'y'} = \pm R \sin(56.87^\circ) \Rightarrow \underline{\tau_{x'y'}(-10^\circ) = \pm 4.19 \text{ ksi}}$$



Example 3:

Given:

The beam shown

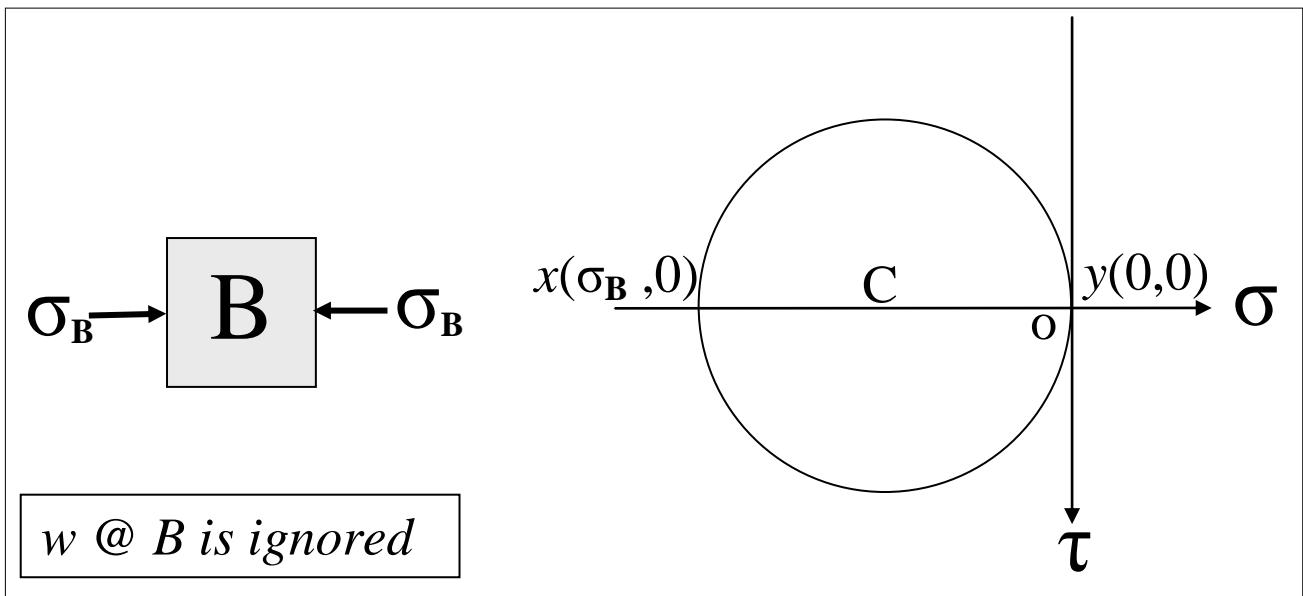
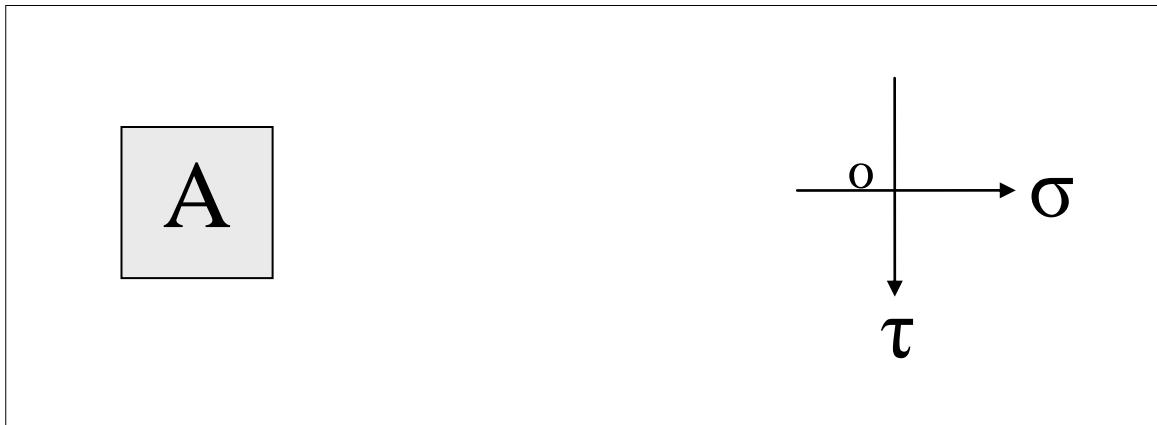


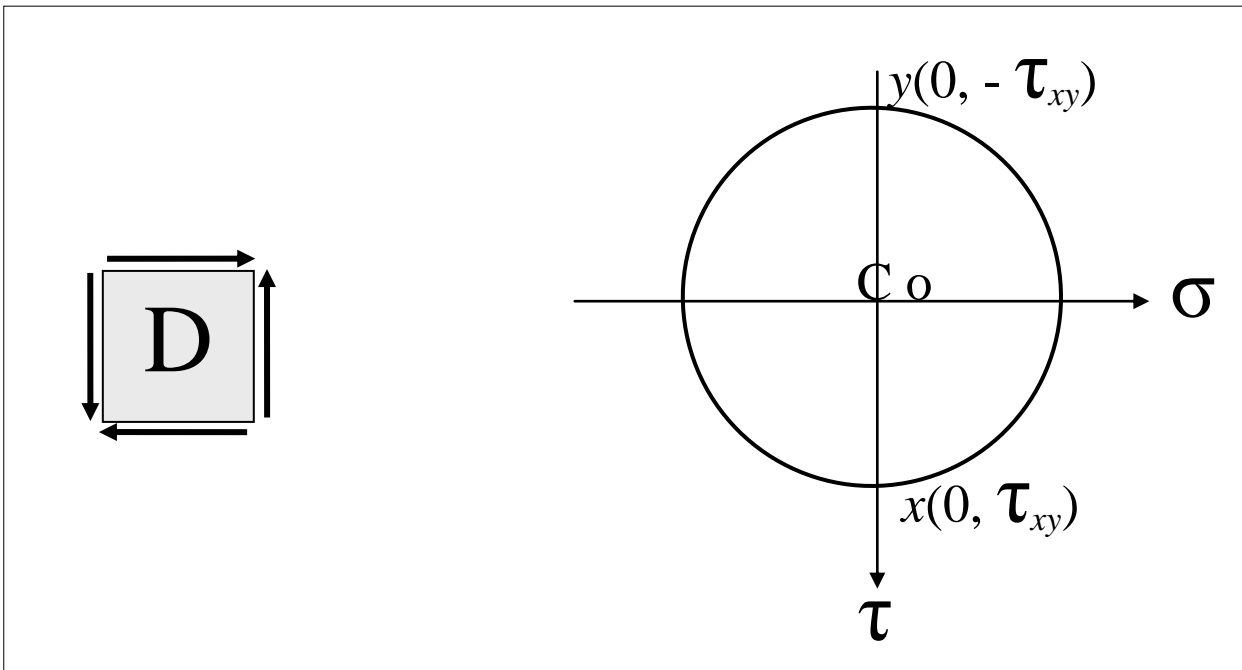
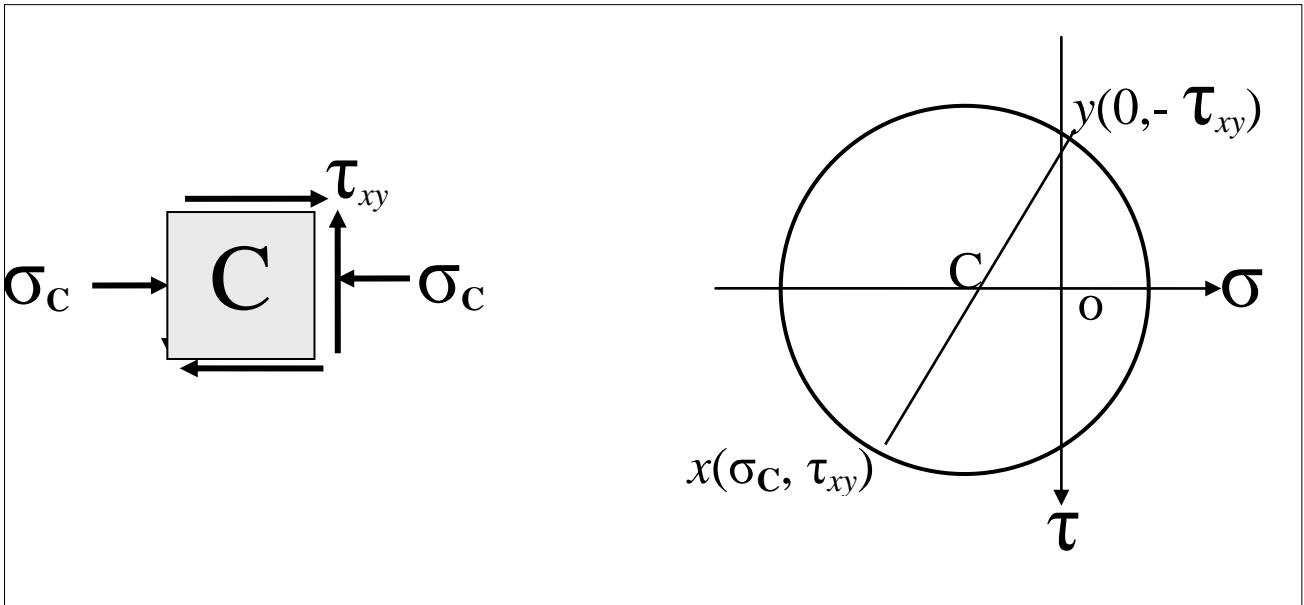
Req'd.:

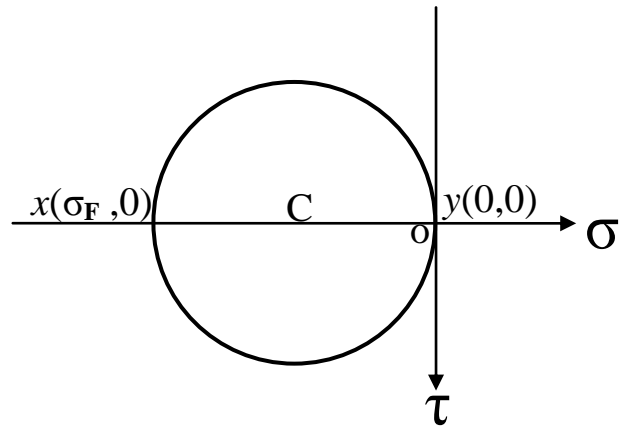
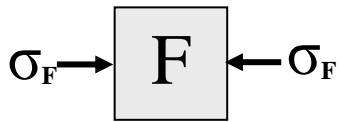
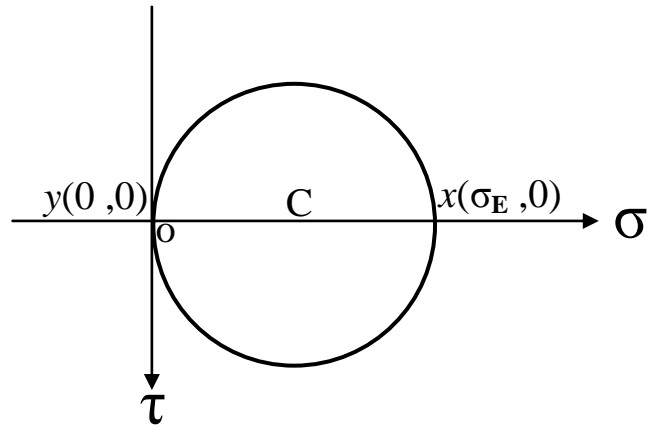
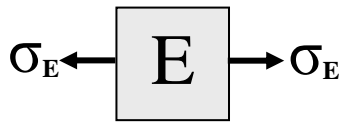
Qualitatively, sketch the **state of stress** and **Mohr's circle** for each of the **points A to G**.

Solution:

Note that σ_y is always assumed **zero** (ignored) in **beams**.





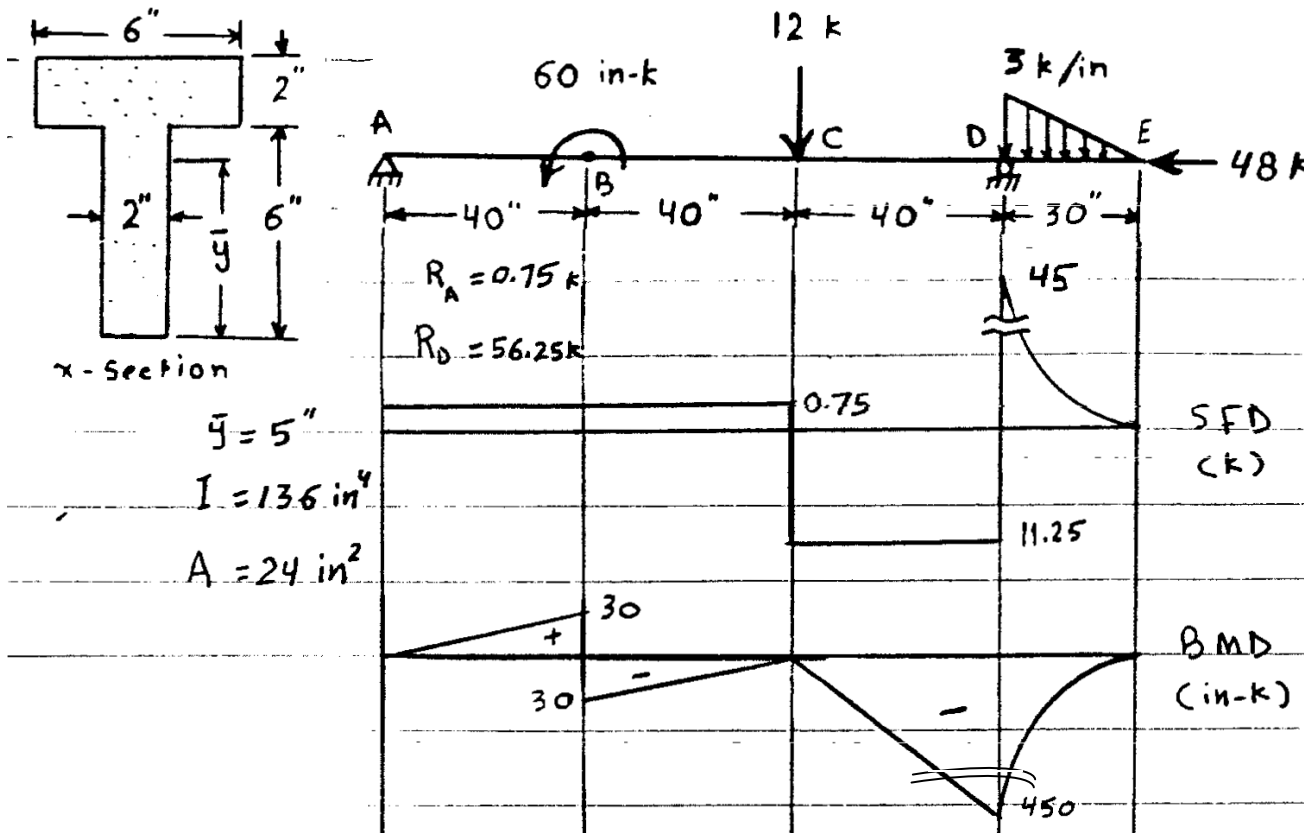


$$G = F \quad \text{Why?!}$$

Example 4:

Given:

The beam shown



Req'd.:

The principal normal and shear stresses and their directions at the point(s) of maximum stresses.

Sol'n.:

$$M_{\max} = 450 \text{ in-k} \quad \curvearrowleft \frac{T}{C} \curvearrowright \quad @ \text{ D}$$

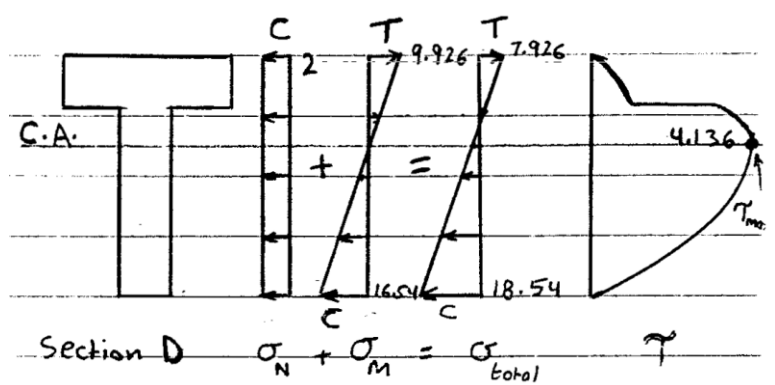
This M will give σ_{\max} (both T & C)

$$V_{\max} = 45 \text{ k} \quad @ \text{ D also}$$

$$\sigma = \sigma_N + \sigma_M = \pm \frac{N}{A} \pm \frac{My}{I}$$

$$\sigma_{\text{top}} = \frac{-48}{24} + \frac{450(3)}{136} = -2 + 9.926$$

$$= 7.926 \text{ ksi (T)}$$



$$\sigma_{\text{bottom}} = -\frac{48}{24} - \frac{450(5)}{136} = -2 - 16.54$$

$$= \mathbf{18.54 \text{ ksi} \quad (C)}$$

$$\sigma_{\text{C.A.}} = \frac{-48}{24} + 0 = 2 \text{ ksi} \quad (C)$$

$$\tau_{\text{top}} = \tau_{\text{bottom}} = 0$$

$$\tau_{\text{C.A.}} = \tau_{\text{max}} = \frac{VQ}{Ib} = 45 (5 \times 2 \times 2.5) / 136(2) = \mathbf{4.136 \text{ ksi}}$$

σ 's above are all σ_x .

$\sigma_y = 0$ at all points (always the case in beam theory)

We need to calculate the principal stresses at 3 points (top, bottom, and N.A. of section D).
 \Rightarrow Choose the maximum normal (T & C) and shear stresses.

1) Top of D : (**Mohr's circle** is used for the max/min stress values below, *not shown*)

$$\sigma_x = 7.926 \text{ ksi} \quad , \quad \sigma_y = 0 \quad , \quad \tau_{xy} = 0$$

$$\Rightarrow \sigma_{\text{max}} = 7.926 \text{ ksi} \quad , \quad \sigma_{\text{min}} = 0 \quad , \quad \tau_{\text{max}} = 3.963 \text{ ksi} \quad , \quad \tau_{\text{min}} = -3.963 \text{ ksi}$$

2) Bottom of D: $\sigma_x = -18.54 \text{ ksi} \quad , \quad \sigma_y = 0 \quad , \quad \tau_{xy} = 0$

$$\Rightarrow \sigma_{\text{max}} = 0 \quad , \quad \sigma_{\text{min}} = -18.54 \text{ ksi} \quad , \quad \tau_{\text{max}} = 9.27 \text{ ksi} \quad , \quad \tau_{\text{min}} = -9.27 \text{ ksi}$$

3) Centroidal Axis: $\sigma_x = -2 \text{ ksi} \quad , \quad \sigma_y = 0 \quad , \quad \tau_{xy} = 4.136 \text{ ksi}$

$$\Rightarrow \sigma_{\text{max}} = 3.2 \text{ ksi} \quad , \quad \sigma_{\text{min}} = -5.2 \text{ ksi} \quad , \quad \tau_{\text{max}} = 4.2 \text{ ksi} \quad , \quad \tau_{\text{min}} = -4.2 \text{ ksi} \quad \Rightarrow$$

| | |
|---|---------------------|
| $\sigma_{\text{max}}^T = 7.926 \text{ ksi}$ | @ D (Top) |
| $\sigma_{\text{max}}^C = 18.54 \text{ ksi}$ | @ D (Bottom) |
| $\tau_{\text{max}} = 9.27 \text{ ksi}$ | @ D (Bottom) |

* **Important note:** When τ_{xy} is zero at a point, then σ_x and σ_y are themselves principal stresses at that particular point (as seen above).