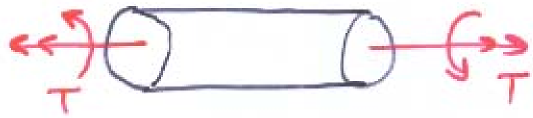
Torsion

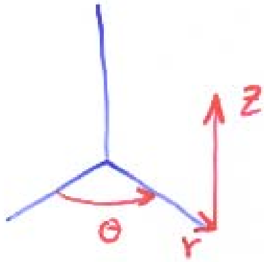
There are many torsionally loaded structural elements in life: in airplanes, automobiles, drill equipments, screw drivers, ...



Kinematics of Circular Shafts

* circular shafts : much simpler, more common, very efficient
  ((You will prove it latter!))

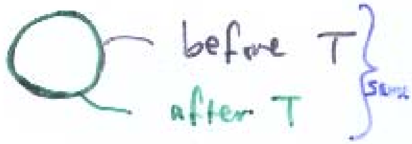
* Cylindrical coordinates system :



* Assumptions :

1) Circular cross sections are rigid

$$\Rightarrow \varepsilon_r = \varepsilon_\theta = \gamma_{r\theta} = 0$$



2) The shaft remains straight and cross-section $\perp$ its geometric axis before deformation remains $\perp$ after deformation. $\Rightarrow$ 90° angles remain 90° $\Rightarrow$ no shear

$$\Rightarrow \gamma_{rz} = 0$$

3) The distance between cross sections does <u>not</u> change.

$$\Rightarrow \varepsilon_z = 0$$

The assumptions above are geometric; they do not depend on the material behavior (elastic or inelastic). However, these assumptions are limited to small deformations.

Summary: $\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0$

Thus, from Hooke's Law:

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$$

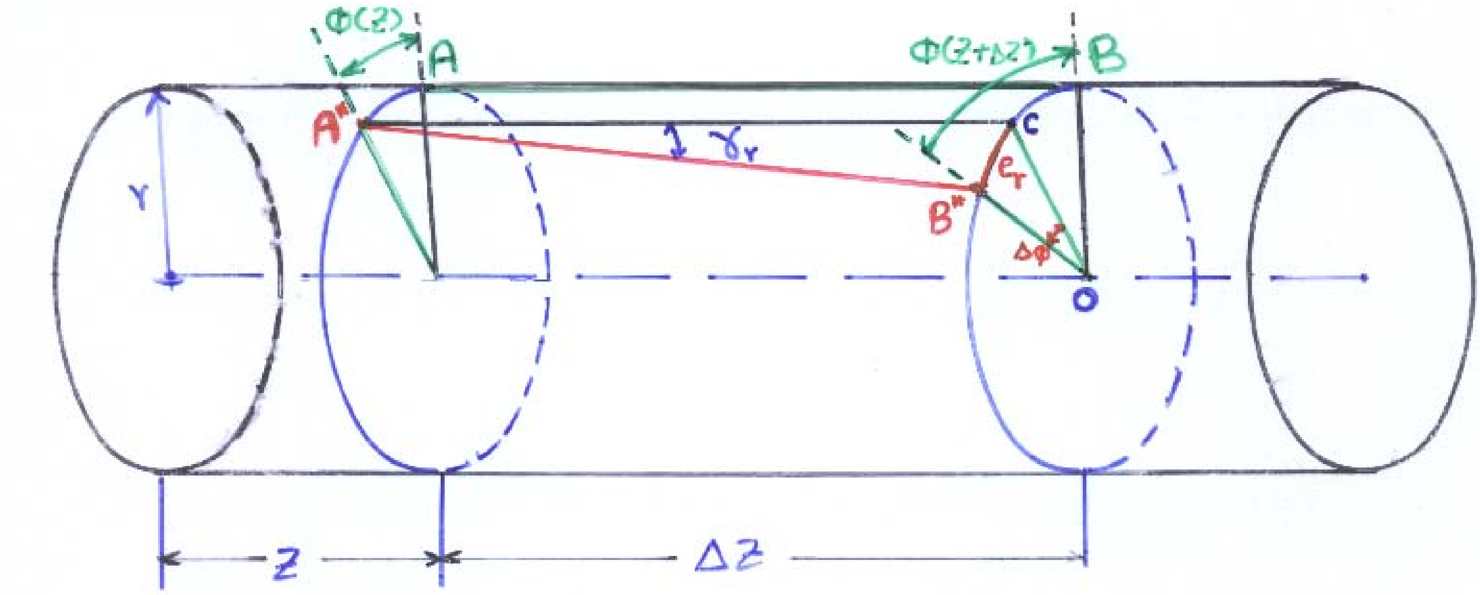
There, we have only

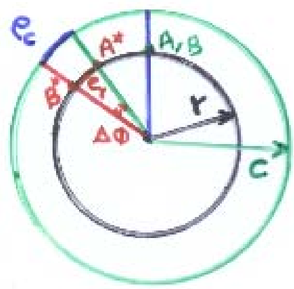
$$\gamma_{\theta z}$$

$$\Rightarrow \quad \tau_{\theta z}$$

Since we have $\gamma_{\theta z}$ and $\tau_{\theta z}$ ($\tau_{z\theta}$) only, we may drop the subscripts. $\Rightarrow$ $\boxed{\gamma}$ and $\boxed{\tau}$

Kinematics:





Very small deformation is assumed $\Rightarrow$

$e_r \approx$ straight line

$e_c \approx$ " "

$\Rightarrow \triangle A^*B^*C$

$\Rightarrow \triangle CB^*O$

Note that e_r is common for the two triangles: A^*B^*C and CB^*O

$$e_r \approx r\Delta\phi$$
$$\approx \gamma_r \Delta z$$

$$\Rightarrow \quad r\Delta\phi = \gamma_r \Delta z \quad \Rightarrow \quad \frac{\Delta\phi}{\Delta z} = \frac{\gamma_r}{r}$$

Let $\Delta \to d \quad \Rightarrow \quad \underline{\underline{\dfrac{d\phi}{dz} = \dfrac{\gamma_r}{r}}}$

The equation above expresses the relative rotation of the cross section at $(z+dz)$ wrt the section at z in terms of the shearing strain at a distance r from the center.

Now, we need to find the angle of twist ϕ and the shearing stress/strain τ/γ.

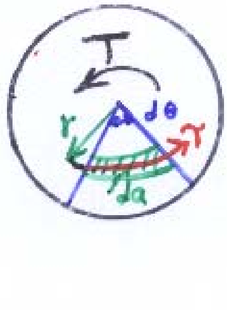
We can not use Statics alone to derive the equations. (TRY!)

Thus, this can be achieved by utilizing:

1. Equilibrium
2. Geometric Compatibility
3. Material Behavior

The problem is <u>internally</u> SI.

<u>Elastic</u> Twisting of <u>Circular</u> Shafts:



1. Equil.:

$$dT = (\tau\, da)\, r$$
$$\int dT = \int (\tau\, da)\, r$$
$$T = \int_A \tau\, r\, da \quad \longleftarrow \text{internal}$$
$$((T = \Sigma M_z)) \quad \longleftarrow \text{external}$$

③ Material Behavior : ⇐ Written before ② G. Comp. as it is short & easy. ⇒ go to ②

$$T = \int \tau r \, da$$
$$= \int G \gamma r \, da$$

② Geometric Compatibility :

$$T = \int G \gamma r \, da$$
$$= \int G \frac{\gamma}{r} r^2 \, da$$
$$= \int G \frac{d\phi}{dz} r^2 \, da$$
$$= G \frac{d\phi}{dz} \int r^2 \, da$$

$\frac{d\phi}{dz}$ is the unit angle of twist

Recall from Statics that $\int r^2 da = J$

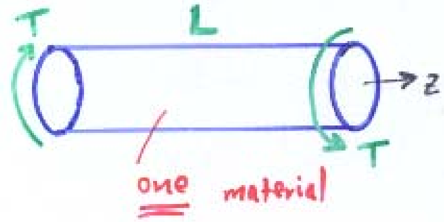
J = polar moment of inertia wrt the z-axis

- $J = \frac{\pi}{2} r^4$ for solid section
- $J = \frac{\pi}{2}(r_{out}^4 - r_{in}^4)$ for hollow section

Thus:

$$\boxed{\frac{d\phi}{dz} = \frac{T}{JG}}$$

For uniform shafts (constant T, J, G) as shown :



$$d\phi = \frac{T}{JG} dz$$
$$\int_{z_i}^{z_{i+1}} d\phi = \int_{z_i}^{z_{i+1}} \frac{T}{JG} dz$$
$$\phi_{i+1} - \phi_i = \frac{T}{JG} \int_{z_i}^{z_{i+1}} dz$$

Thus, for a uniform shaft :

$$\boxed{\phi = \frac{TL}{JG}}$$ ⇐ total angle of twist

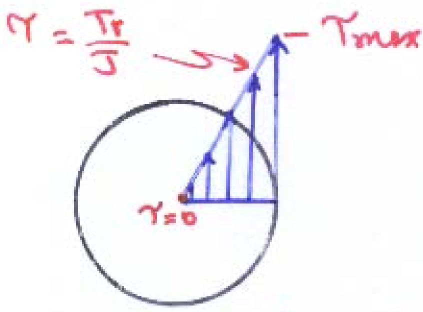
For nonuniform shafts, $\frac{T}{JG}$ may contain discontinuities caused by abrupt changes in the cross section, the applied torques and/or in the material.

The three methods of analysis [① direct integration, ② discrete element, and ③ superposition] discussed earlier in axially-loaded members problems can be used here.

$$\underline{\underline{\Phi_{total} = \sum \Phi_i = \sum \frac{TL}{JG}}}$$

Shearing Stress (τ):

$$\tau = G\gamma$$
$$= G\frac{\gamma}{r}r$$
$$= G\frac{d\Phi}{dz}r$$
$$= G\left(\frac{T}{JG}\right)r$$



$\Rightarrow$ $$\boxed{\tau = \frac{Tr}{J}}$$ $\Leftarrow$ r is <u>any</u> radius in the shaft

$$\boxed{\tau_{max} = \frac{T r_{max}}{J}}$$ $\Leftarrow$ r_{max} is usually the outside radius r_{out}

$\tau = 0$ @ $r = 0$

$\Uparrow$

This formula gives τ_{max} @ r_{max} in the <u>same</u> section.

τ_{max} may also be at other sections with <u>T_{max}</u> and/or <u>J_{min}</u>.