Problem 1: Key Solution

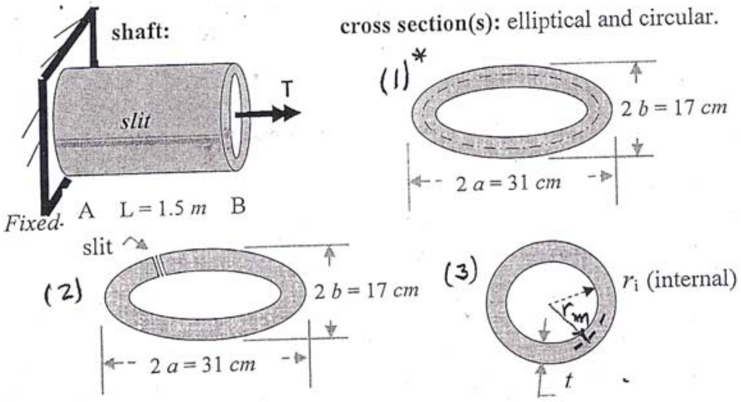
Cross section of a tubular shaft AB may be selected as a **closed** *elliptical thin-walled cross section*, or as **open** *elliptical thin-walled* (with a longitudinal slit (*cut*) shown in Fig. P-1 below), or as **circular** cross-section. For a given torque T = 76 kN.m:

a) Compute and compare **angular twists** ϕ_B for the two elliptical cross-sections. (10) pts.

b) Compute the **ratio of maximum shear stresses** R_τ for the closed **elliptical section** and the **circular cross-section** (*namely*: $R_\tau = \frac{\tau_{circular}}{\tau_{elliptical}}$). (10) pts.

c) Use value of R_τ (computed above) to **select the most efficient** closed cross-section (**elliptical** or **circular**) to carry torque T. (5) pts.

Fig. P-1: Cross-sections of equal areas; uniform $t = 1$ cm for all cross sections; G = 80 GPa.



Note:

For ellipse: *area* $A = \pi ab$; and *circumference* $S_m = \pi\left[1.5(a+b) - \sqrt{ab}\right]$

Note* Cross sections are numbered as (1), (2) and (3).

Geometry Analysis ⇒

$$A_1 = \pi[a_e b_e - a_i b_i] = 72.26 \text{ cm}^2 \qquad \because A_1 = A_3 \Rightarrow r_i = \frac{A_1 - \pi t^2}{2\pi t}$$

$$r_i = 11 \text{ cm}$$

$$A_{m(1)} = \pi a_m b_m = \pi (15)(8) \cong 377.0 \text{ cm}^2 \ ; S_m \simeq 73.97 \text{ cm}$$

$$A_{m(3)} = \pi r_m^2 = \pi (11.5)^2 = 415.5 \text{ cm}^2$$

$$J_{eq(2)} = \tfrac{1}{3} S_m t^3 \simeq 24.66 \text{ cm}^4$$

a) $$\phi_{B(1)} = \phi_{B/A} = \left(\frac{Tl}{4GA_m^2}\,\frac{S_m}{t}\right)_1 = 76,(1.5)\left(\frac{73.97}{1.0}\right)\Big/4\times 80\times10^6 (0.0377)^2$$

$$= 1.8542\times10^{-2} \text{ rad.}$$

$$\phi_{B(2)} = Tl/GJ_{eq} = 76.\times 1.5/80\times10^6(2.466\times10^{-7}) = 5.78 \text{ rad}$$

$$R_\phi = \phi_{B(2)}/\phi_{B(1)} = 5.78/1.8542\times10^{-2} \simeq 312.$$

b) Stress ratio $$R_\tau = \tau_{(3)}/\tau_{(1)} = (T/2A_m t)_{(3)}\big/(T/2A_m t)_{(1)} = a_m b_m / r_m^2$$

$$= (30/2)(16/2)/(11.5)^2 = 0.907$$

c) Since $\tau_{(3)}^{max} \cong 91\%\ \tau_{(1)}^{max}$, then circular section is most efficient.

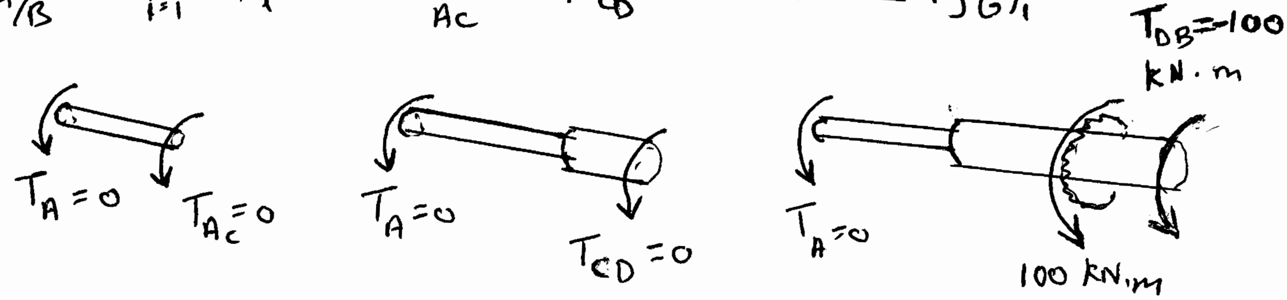
Prob. #2

Method ①

First check the angular gap if it closes or not.

(pts.) Assume it does not close. (Why ?!) ⇒

$$\Phi_{A/B} = \sum_{i=1}^{3} \Phi_i = \Phi_{AC} + \Phi_{CD} + \Phi_{DB} = \Sigma\left(\frac{TL}{JG}\right)_i$$

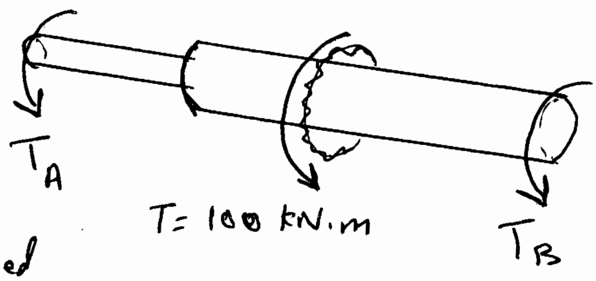
④ 

$$|\Phi_{A/B}| = 0 + 0 + \left|\frac{-100(10)^3(4)}{2.5(10)^{-4}\pi(50)(10)^9}\right| = 0.0102 \text{ rad.} > 0.006$$

⇒ The angular gap closes. ⇒ There is a reaction at A.

① Note that $J_{CD} = J_{DB} = \frac{\pi}{2}\left[(0.15)^4 - (0.05)^4\right] = 2.5(10)^{-4}\pi \simeq 7.85398(10)^{-4}\ m^4$

② FBD: Note that the two reactions are assumed positive. They can be assumed in the other direction as we know they will be so due to T.



② Equilibrium Eq.: $\Sigma T = 0 \Rightarrow T_A + T_B + 100(10)^3 = 0$

One eq. only and two unknowns ⇒ Stat. Indet. ⇒

We need to get one eq. from geom. compat. ⇒

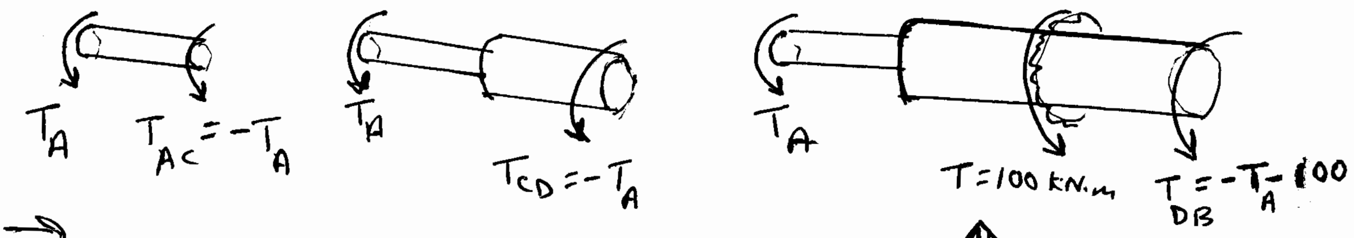
④ $$\sum_{i=1}^{3} \Phi_i = \Sigma\left(\frac{TL}{JG}\right)_i = \Phi_{AC} + \Phi_{CD} + \Phi_{DB} = -0.006$$

Note the minus sign (due to the direction of the applied torque and the rotation of the left end).

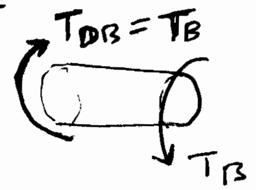
① $J_{AC} = \frac{\pi}{2}(0.1)^4 = 5(10)^{-5}\pi \simeq 1.5708(10)^{-4}\ m^4$

Prob. #2

FBD

(3) 

⇕ or



⇒

$$\frac{-T_A\,(2)}{5(10)^{-5}\pi(50)(10)^9} + \frac{-T_A\,(3)}{2.5(10)^{-4}\pi(50)(10)^9} + \frac{[-T_A - 100(10)^3](4)}{2.5(10)^{-4}\pi(50)(10)^9} = -0.006$$

⇒ $-4.329011(10)^{-7}\,T_A = 4.18592(10)^{-3}$ ⇒

(2) $T_A = -9669.5$ N.m ⇒ $T_A = 9.6695$ kN.m in the opp. dir. (T_A)

(1) From equil. eq., $T_B = -90330.5$ N.m $= 90.331$ kN.m (T_B)

$\tau = \frac{Tr}{J} = \frac{T\rho_{max}}{J} = \frac{T\,r_{max}}{J}$ ⇒

We need to check τ_{max} in AC and D.B only. (Why?!)

(2) $\tau^{AB}_{max} = \frac{9669.5\,(0.1)}{5(10)^{-5}\,\pi} = 6.156$ MPa @ outer radius

(2) $\tau^{DB}_{max} = \frac{90331\,(0.15)}{2.5(10)^{-4}\,\pi} = 17.25$ MPa @ outer radius

(1) ⇒ $\boxed{\tau_{max} = 17.25 \text{ MPa @ the outer radius of DB}}$

method (2)

See me if interested.

Prob. 2

Method ②: (Optional)

Ignoring the support at A, let's calculate how much A rotates (freely) due to $T = 100$ kN.m @ D. ⇒

$$|\Phi^{free}_{A/B}| = \sum_{i=1}^{3} \Phi_i = \Phi_{AC} + \Phi_{CD} + \Phi_{DB} = \sum \left(\frac{TL}{JG}\right)_i$$

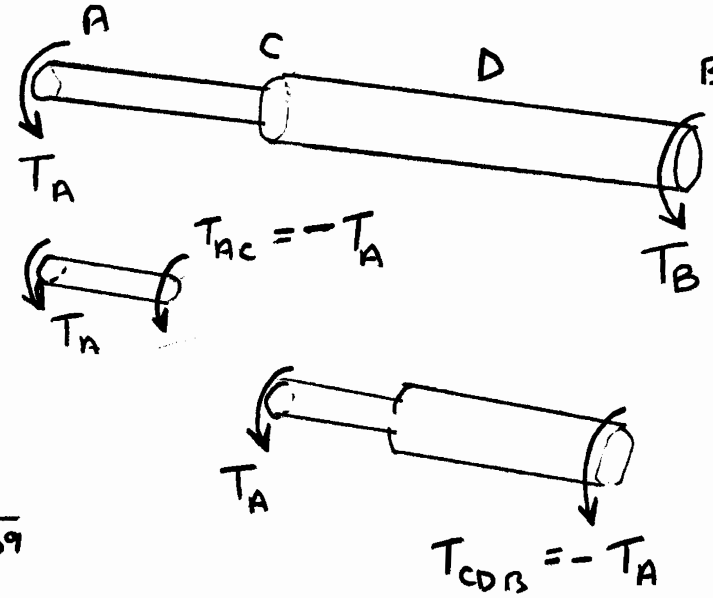
$$= 0 + 0 + \frac{-100(10)^3(4)}{2.5(10)^{-4}\pi(50)(10)^9} = 0.010186 \text{ rad}$$

A is free to rotate by ± 0.006 rad ⇒

$$\Phi_A^{prevented} = 0.010186 - 0.006 = -4.18592(10)^{-3} \text{ rad.}$$

Thus, T_A needed to be applied to prevent $\Phi_A^{prev.}$ ⇒

Note: no T at D (Why?!)



$$\Phi_{A/B} = \Phi_{AB} + \Phi_{CDB}$$

$$= \frac{-T_A(2)}{5(10)^{-5}\pi(50)(10)^9} + \frac{-T_A(7)}{2.5(10)^{-4}\pi(50)(10)^9} \equiv 4.18592(10)^{-3} \Rightarrow$$

$$T_A = -9669.5 \text{ N.m} = 9.6695 \text{ kN.m}$$

From equil.: $T_A + T_B + 100 = 0 \Rightarrow T_B = -90.331$ (in orig. sketch) $= 90.331$ kN.

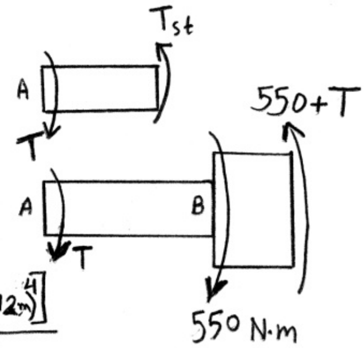
Thus, $\tau^{AB}_{max} = \frac{9669.5(0.1)}{5(10)^{-5}\pi} = 6.159$ MPa @ outer radius

$\tau^{DB}_{max} = \frac{90331(0.15)}{2.5(10)^{-4}\pi} = 17.25$ MPa @ outer radius

⇒ $\tau_{max} = 17.25$ MPa @ the outer radius of DB

Problem 3:

Problem 2:



- For the steel shaft:

The internal torque = T

$$\tau_{max} = (\tau_{all})_{st} = \frac{T \cdot C_{st}}{J_{st}}$$

$$\Rightarrow T_{max} = \frac{\tau_{all} \cdot J_{st}}{C_{st}} = \frac{(90 \times 10^6 Pa) \cdot \left[\frac{\pi}{2}(0.012m)^4\right]}{(0.012m)}$$

$$\Rightarrow \underline{T_{max} = 244.29\ N \cdot m}$$

- For the Aluminum shaft:

The internal torque = 550 + T

$$\tau_{max} = (\tau_{all})_{Al} = \frac{(550+T) \cdot C_{Al}}{J_{Al}} \Rightarrow 550+T = \frac{(60 \times 10^6 Pa) \cdot \left[\frac{\pi}{2} \cdot (0.02m)^4\right]}{(0.02m)}$$

$$\Rightarrow 550 + T = 753.98\ N \cdot m \Rightarrow \underline{T_{max} = 203.98\ N \cdot m}.$$

- For the max. allowable angle of twist, $\phi_{A/C} = 0.05$ rad:

$$\phi_{A/C} = \phi_{A/B} + \phi_{B/C}$$

$$\Rightarrow 0.05\ rad = \frac{T \cdot (1\ m)}{\frac{\pi}{2}(0.012\ m)^4 \cdot 75 \times 10^9\ Pa} + \frac{(550+T) \cdot (1\ m)}{\frac{\pi}{2}(0.02\ m)^4 \cdot 26 \times 10^9\ Pa}$$

$$\Rightarrow 0.05\ rad = \frac{T}{2442.90} + \frac{(550+T)}{6534.51}$$

$$\Rightarrow 122.15 = T + 0.3738(550+T)$$

$$\Rightarrow \underline{T = -60.74\ N \cdot m}$$ (opposite direction).

$\therefore$ The largest torque, $T = 60.74\ N \cdot m$

Problem 4: Draw the SFD & BMD using the graphical (Areas summation) method.