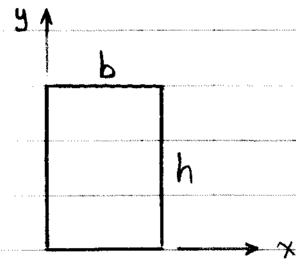
#25

Examples
Moments of Inertia
Integration Method

Example 1 :



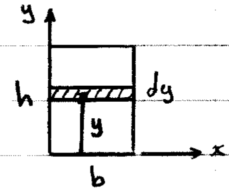
Given :

The rectangle shown

Req.d. :

I_x by a) horizontal differential element

b) vertical " "

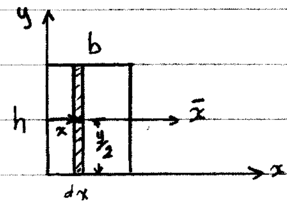


Sol.n. :

$I_x = \int y^2 \, dA$

a) horizontal element : $dA = b\,dy$

$\Rightarrow I_x = \int_0^h y^2 \, b \, dy = b \int_0^h y^2 \, dy \Rightarrow \boxed{I_x = \frac{1}{3} b h^3}$



b) vertical element :

$I_x = \int y^2 \, dA = \int dI_x$

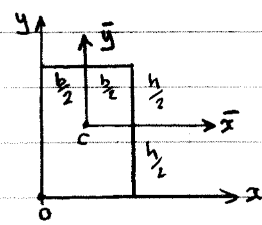
$dI_x = d\bar{I}_{\bar{x}} + Ad^2 = (\frac{1}{12} dx \, h^3) + (dx \, h)(\frac{h}{2})^2$

$= \frac{1}{3} h^3 dx$

$I_x = \int dI_x = \int_0^b \frac{1}{3} h^3 dx \Rightarrow \boxed{I_x = \frac{1}{3} b h^3}$ as above } Compare the two methods!

(Similarly, I_y can be found)

Example 2 :



Given:

The rectangle shown

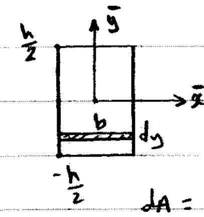
Req.d. :

$\bar{I}_{\bar{x}}$ by a) integration

b) using parallel axis theorem and the result of Example 1 above

Sol.n. :

a) $\bar{I}_{\bar{x}} = \int \bar{y}^2 \, dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{y}^2 (b\,dy) = b (\frac{\bar{y}^3}{3}) \Big|_{-\frac{h}{2}}^{\frac{h}{2}}$



$= \frac{1}{3} b \left[(\frac{h}{2})^3 - (\frac{-h}{2})^3 \right] = \frac{1}{3} b (\frac{h^3}{4}) \Rightarrow \boxed{\bar{I}_{\bar{x}} = \frac{1}{12} b h^3}$

$dA = b\,dy$

b) $I_x = \bar{I}_{\bar{x}} + Ad^2 \Rightarrow$

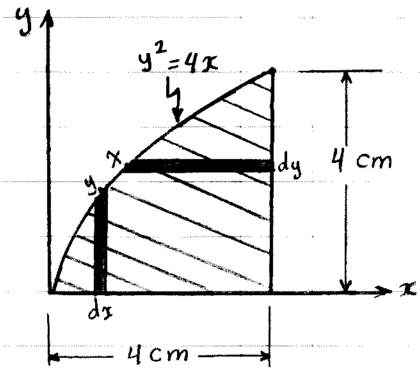
$\bar{I}_{\bar{x}} = I_x - Ad^2 = \frac{1}{3} b h^3 - (bh)(\frac{h}{2})^2 = \frac{1}{3} b h^3 - \frac{1}{4} b h^3 = \frac{1}{12} b h^3$ as above

* important : Note the factor $\frac{1}{3}$ in I_x and $\frac{1}{12}$ in $\bar{I}_{\bar{x}}$ (I about centroidal axis).

Example 3 :

Given :
The shaded area shown

Req'd. :
I_x and I_y



Sol.n. :

$I_x = \int y^2 \, dA$

For I_x it is easier to take a horizontal element (Why? Try vertical element!)

$dA = (4 - x)\, dy \Rightarrow$

$$I_x = \int y^2 (4 - x)\, dy$$
$$= \int y^2 \left(4 - \frac{y^2}{4}\right) dy = \int_0^4 \left(4y^2 - \frac{1}{4} y^4\right) dy$$
$$= \left[\frac{4}{3} y^3 - \frac{1}{20} y^5\right]_0^4 \Rightarrow \boxed{I_x = 34.1 \text{ cm}^4}$$ ← watch the units

Now, consider a vertical element :

$dI_x = \frac{1}{3} (dx)\, y^3$ ← from the previous example

$$I_x = \int \frac{1}{3} y^3 dx = \frac{1}{3} \int_0^4 (4x)^{\frac{3}{2}} dx$$
$$= \frac{1}{3} \left(\frac{1}{4}\right)\left(\frac{2}{5}\right)(4x)^{\frac{5}{2}} \Big|_0^4 = 34.1 \text{ cm}^4$$ as above

$$I_y = \int x^2 \, dA$$

Take a vertical strip (element) $\Rightarrow$

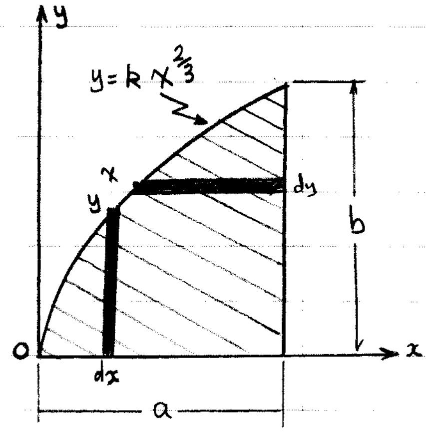
$dA = y\, dx$

$$I_y = \int x^2 y\, dx = \int_0^4 x^2 (2\sqrt{x})\, dx$$
$$= \int_0^4 2 x^{\frac{5}{2}} dx = 2\left(\frac{2}{7}\right)(4)^{\frac{7}{2}} \Rightarrow \boxed{I_y = 73.1 \text{ cm}^4}$$

* Recalculate I_y by taking a horizontal element.

$I_y > I_x$, Why?

Example 4:



Given:

The shaded area shown

Req.d.:

I_x, k_x, I_y, k_y, J_o, k_o

Soln.:

$I_x = \int y^2 dA$

$dA = (a-x)\,dy$ (horizontal element)

$I_x = \int y^2 (a-x)\,dy$

$= \int_0^b y^2 (a - \frac{1}{k^{\frac{3}{2}}} y^{\frac{3}{2}})\,dy = \int_0^b (ay^2 - \frac{1}{k^{\frac{3}{2}}} y^{\frac{7}{2}})\,dy$ From B.C., $k = b/a^{\frac{2}{3}}$

$= \frac{a}{3} b^3 - \frac{2}{9}\left(\frac{b^{\frac{9}{2}}}{(b/a^{\frac{2}{3}})^{\frac{3}{2}}}\right) = \frac{1}{3} ab^3 - \frac{2}{9} ab^3 \Rightarrow \boxed{I_x = \frac{1}{9} ab^3}$

Try a vertical element.

$k_x = \sqrt{I_x / A}$

$dA = (a-x)dy \Rightarrow A = \int (a-x)dy = \int_0^b [a - (\frac{y}{k})^{\frac{3}{2}}]\,dy = \frac{3}{5} ab$

$\Rightarrow k_x = \sqrt{\frac{1}{9} ab^3 / \frac{3}{5} ab} \Rightarrow \boxed{k_x = \sqrt{5/27}\ b}$

$I_y = \int x^2 dA$

$dA = y\,dx$ (vertical element; try a horizontal one.)

$I_y = \int x^2 y\,dx = \int_0^a k x^{\frac{8}{3}}\,dx = k(\frac{3}{11}) a^{\frac{11}{3}} \Rightarrow \boxed{I_y = \frac{3}{11} a^3 b}$

$k_y = \sqrt{I_y / A} = \sqrt{\frac{3}{11} a^3 b / \frac{3}{5} ab} \Rightarrow \boxed{k_y = \sqrt{5/11}\ a}$

$J_o = \int r^2 dA = I_x + I_y$

$= \frac{1}{9} ab^3 + \frac{3}{11} a^3 b \Rightarrow \boxed{J_o = \frac{38}{99} ab(a^2 + b^2)}$

$k_o = \sqrt{J_o / A}$

$= \sqrt{\frac{38}{99} ab(a^2+b^2) / \frac{3}{5} ab} \Rightarrow \boxed{k_o = \sqrt{\frac{190}{297}(a^2+b^2)}}$