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On the frequency dependence of the modulus of elasticity of wood

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Abstract This short note reviews the reasons for the frequency dependence of the Modulus of Elasticity, MOE, of wood. It has in fact been reported in several publications on wood that depending on the technique used in the test experiment, the value of the MOE depends to some degree on the frequency at which it is evaluated. The frequency ranges used are namely zero frequency in the case of static bending, audio frequencies when using mechanical vibrations or sound radiation and finally ultrasonics. The results from implementing these three different techniques show that the lowest value that may be obtained for the MOE occurs when using the static mode, and thereafter increases with increasing frequency. This property of increasing dynamic MOE with frequency is shared by all solid materials, and finds its theoretical explanation in the Kramers-Kronig relations. Dispersion in conjunction with the notion of complex MOE permit to establish the relation between the real and the imaginary components of the MOE, i.e. respectively the dynamic and loss moduli. Due to the mathematical difficulties encountered in using the exact expressions, approximations are necessary for applications in practical situations. Hence, an improved version of the Zener model for viscoelasticity, which has lately been proposed by Pritz (1999), is presented. With some assumptions, and under which excellent agreement has been obtained with the exact theory, this model is used for predicting the viscoelastic properties of wood.

Introduction

The outstanding properties of wood make this natural material the subject of interest of many researchers. Not only that its physical characteristics, like inhomogeneity and anisotropy interest engineers and material scientists, but wood-related industries are also concerned about other properties of this material, especially its strength. Regarding this latter property, it has frequently been reported in the literature that the mechanical properties of wood as assessed by means of dynamical testing techniques are to some extend dependent on the frequency of operation. As mentioned by Hearmon (1966), the choice of the method for determining the Modulus of Elasticity is often a matter of convenience, but he points out at the same time that the correlation between the static MOE and the dynamic one is not always perfect, and refers to the results of Jayne

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(1959). These conclusions have also been shared by other researchers (Burmester 1965; Marra et al. 1966) and are supported by more recent investigations (Haines et al. 1996, and references therein; Olsson and Perstorper 1992) although others, for known or unknown reasons have observed no noticeable variations of the value of the MOE with frequency (Fukada 1950). Regarding the use of ultrasound as a means for assessing the properties of wood, the values of the MOE as measured from the speed of propagation of the ultrasonic signal have also been found to exceed those of the static MOE (Bucur 1983; Bucur and Feeney 1992; Halabe et al. 1996). This trend of increasing value of the MOE with the frequency is indeed a property found in all solid materials, and is a consequence of applying simple, but not widely known, mathematical concepts to the study of linear causal systems. It is henceforth the goal of this note to shed some light on the causes underlying such frequency-dependent behavior.

The notion of complex modulus of elasticity

In general, solid materials are known to have both elastic and damping properties. Solid materials are not ideally elastic, and consequently, for a time varying stress, the stress and strain are not always in phase. Solid materials need in a way some time to react fully to some action, and therefore exhibit a memory-like behavior; the response to some action does not only depend on time, but also on past history. Instead of pure elasticity, one would rather talk about the visco-elasticity of these materials.

If $\sigma(\omega)$ and $\varepsilon(\omega)$ are, respectively, the stress and strain spectra of a material sample, then according to Hooke's law, the Modulus of Elasticity, MOE and symbolised by $M(\omega)$, is defined by:

$$\sigma(\omega) = M(\omega) \cdot \varepsilon(\omega) \tag{1}$$

A powerful means to studying the properties of materials is the notion of complex MOE, where the MOE may be written as the sum of a real part, often denoted as the dynamical modulus, and an imaginary part, the loss modulus, i.e.

$$M(\omega) = \frac{\sigma(\omega)}{\varepsilon(\omega)} = M_d(\omega) + jM_l(\omega) = M_d(\omega)(1 + j\eta(\omega)) \tag{2}$$

with the real part $M_d(\omega)$ denoting the dynamical modulus, and the imaginary part $M_l(\omega)$, denoting the loss modulus and

$$\eta(\omega) = \frac{M_l(\omega)}{M_d(\omega)} \tag{3}$$

a coefficient used for assessing the amount of loss in the material and by way called the loss factor.

Dispersion of materials and response of linear systems

The name of dispersion is taken from the phenomenon of optical dispersion. It is known that the index of refraction n of a transparent medium is dependent on the frequency of the light falling on it. This results in the common observation that light rays of different colours, i.e. of different wavelengths or frequencies, are deviated at different angles at their passage through a transparent glass prism.

This may be explained by the fact that n is expressed by a real part determined by the phase velocity of light through the medium, and an imaginary part taking into account the absorption of light in it. Therefore, a similarity may be drawn from comparing the index of refraction in optics and the MOE in mechanics, and consequently solid materials are also said to be dispersive in that they may respond differently toward excitations with different frequencies. It goes back to Kronig (1926) and Kramers (1927) who were the first to introduce the concept of dispersion relations when they were able to show that the real part of (n^2-1) could be expressed as an integral of the imaginary part of the same quantity. But a general label of dispersion relations would be to consider any pair of equations expressing the real part of a function as an integral of its imaginary part and vice versa.

Another important feature about real solid materials, or more generally all real physical systems, is that they are causal, i.e. no response is expected from the sample of material or the system under test prior to the application of any excitation. This leads to the fact that the so-called Impulse Response, and likewise the frequency response function, which is simply the frequency form of the impulse response of the material sample or of the system, are one-sided functions, and are therefore zero at times earlier than that of application of the excitation. In linear systems theory, the principle of causality has for main consequence that the real and imaginary parts of the frequency response function are interrelated (see for instance Hahn 1996, p. 286 or Papoulis 1962 p. 198). It is important to note at this point that from theoretical considerations nothing remarkable may be said about the frequency dependencies, except perhaps that in the case of the MOE the loss modulus is zero at zero frequency; i.e. no motion, no energy loss (Pritz 1998). Henceforth, the dispersion relations may be taken as a consequence of causality, and are therefore independent of the details of the particular interaction described by the Impulse Response or the frequency response function; the dispersion relations are derivable by means of application of the Cauchy integral formula for the Modulus of Elasticity which is a complex frequency function (Arfken 1985; Booij and Thoone 1982).

Mathematical formulations of material dispersion and consequences

In the mathematical context, the dispersion relations are formulated with the help of Hilbert transforms, and are of general nature, finding applications in several branches of physics, including acoustics, electromagnetism and optics. Several forms of such pairs of relations have been formulated for viscoelastic materials, and, for instance, a simplified form of such a set which includes the static Modulus M_0 is (Tschoegel 1989):

$$M_d(\omega) = M_0 + \frac{2\omega^2}{\pi} \mathrm{P} \int_0^\infty \frac{M_l(x)/x}{\omega^2 - x^2} dx; \quad M_l(\omega) = -\frac{2\omega}{\pi} \mathrm{P} \int_0^\infty \frac{M_d(x)}{\omega^2 - x^2} dx \qquad (4)$$

where x is an integration variable and P stands for the principal value of the integrals. These last formulas express the fact that the knowledge of the frequency behavior of one of the moduli permits the determination of the other modulus at any frequency. Usually, the study of a material sample requires ideally the knowledge of its response to a Dirac impulse-like excitation, but the response to a step excitation is also a good alternative. Actually, the step excitation is more easily realisable in practical experimental situations, and the Dirac delta and the unit step functions are not completely unrelated as a Fourier transform exists between the two, see for instance (Arfken 1985). A further requirement for deriving the dispersion relations is that the response has no singularities at the origin of time. The mathematical consequence of this condition is that the frequency functions of the moduli satisfy some simple properties regarding parity, namely that:

$$M_d(\omega) = M_d(-\omega) \quad \text{and} \quad M_l(\omega) = -M_l(-\omega) \tag{5}$$

meaning that the dynamic modulus is an even function whereas the loss modulus is an odd function of frequency. The curve of the loss modulus must therefore go through zero at zero frequency. For the dynamic modulus, its frequency dependency being symmetric may suggest that the curve has a zero slope at zero frequency, this is however true only on a log-log scale, and the slope may be different from zero on a lin-lin scale. The value of the dynamic modulus at zero frequency is equal to that of the static modulus.

To study in some more detail the overall frequency dependence of the moduli, and also for practical applications, one needs to reformulate the general dispersion relations in simpler and more suitable forms. Hence, for moduli not behaving in a resonance-like manner, a restriction to limited frequency ranges sometimes permits to obtain a satisfactory estimation of one modulus from knowledge of the local slope of the other modulus, for instance that (O'Donell et al. 1981):

$$M_l(\omega) \approx \frac{\pi}{2}\omega\frac{dM_d(\omega)}{d\omega} \text{ or } M_l(\omega) \approx \frac{\pi}{4.6}\omega\frac{dM_d(\omega)}{d[\log\omega]} \text{ and } \eta(\omega) \approx \frac{\pi}{2}\frac{d[\log M_d(\omega)]}{d[\log\omega]} \tag{6a, b, c}$$

Focusing our interest on the behavior of the dynamic modulus with frequency, one needs first to express the loss modulus in a general form that may help us draw some important conclusions from Eqs. (6). An alternative expression of the loss modulus may be given by (Cremer and Heckl 1988):

$$M_l(\omega) = \frac{W_l(\omega)}{\pi\eta\varepsilon^2} \tag{7}$$

where ε is the strain amplitude and W_l the energy lost during one period of vibration. For physical real materials, all these quantities are positive. Therefore, M_l, and thus the slope of the dynamic modulus, must, according to the expression in (6a), also be positive. We reach then to the important conclusion that the dynamic modulus is an increasing function of the frequency. The slope of the frequency curve may vary with the frequency, but this variation diminishes with increasing frequencies, and in the limit of infinite frequency the dynamic modulus tends to a finite value which may be denote by M_∞. Note also from the approximate expression in (6a) that the loss modulus, and thereby the amount of damping, is proportional to both the frequency and the slope of the frequency curve of the dynamic modulus at the corresponding frequency. The steeper the curve of the dynamic modulus the more pronounced will be the damping. This characteristic may be used to differentiate between materials with different damping capacities. For wood, wood based materials and cork, the values of the loss factor lie typically in the range between around 0.01 and 0.1.

So far, the analysis was mostly of a qualitative character. To get a quantitative view of these implications, and due to mathematical difficulties, one must resort to approximations of the exact formulas. The Kramers-Kronig relations are then substituted by local versions, which permit, for instance, to estimate the damping properties at a single frequency from knowledge of the frequency variation of the dynamic modulus. These approximations are however valid only under the assumption that the dynamic modulus is a slowly varying function of the frequency, a property found in several solid materials, but not necessarily in wood. Nevertheless, Pritz used a new approach to solving viscoelastic problems. The starting point is from the simplest model, namely the Zener model, also known as the standard viscoelastic body (Zener 1948), and which is known to fail to describe adequately viscoelastic problems. Hence, an improvement was suggested to this model through using it in conjunction with the concept of fractional derivates (Pritz 1996; Torvik and Bagley 1984). The model was consequently named by its author as the fractional Zener model and, accordingly, the MOE is expressed as:

$$M(\omega) = \frac{M_0 + M_\infty (j\omega\tau_r)^\alpha}{1 + (j\omega\tau_r)^\alpha} \tag{8}$$

and an identification of the real and imaginary parts as respectively the dynamic and loss moduli gives in the normalised form:

$$\frac{M_d(\omega)}{M_0} = \frac{1 + (c+1)\cos(\alpha\pi/2)\omega_n^\alpha + c\omega_n^{2\alpha}}{1 + 2\cos(\alpha\pi/2)\omega_n^\alpha + \omega_n^{2\alpha}} \tag{9a}$$

$$\frac{M_l(\omega)}{M_0} = \frac{(c-1)\sin(\alpha\pi/2)\omega_n^\alpha}{1 + 2\cos(\alpha\pi/2)\omega_n^\alpha + \omega_n^{2\alpha}} \tag{9b}$$

The loss factor is then simply the ratio of these two expressions, i.e.

$$\eta(\omega) = \frac{(c-1)\sin(\alpha\pi/2)\omega_n^\alpha}{1 + (c+1)\cos(\alpha\pi/2)\omega_n^\alpha + c\omega_n^{2\alpha}} \tag{10}$$

In these last equations, α is the order of the fractional derivative and is such that $0 < \alpha < 1$, $c = M_\infty / M_0$, and $\omega_n = \omega\tau_r$ is the normalised frequency, τ_r being the relaxation time. Further approximations may be processed at very low frequencies. The model just described is also known sometimes as the four-parameter fractional derivative model, after the number of parameters involved (M_0, M_∞, τ_r and α).

Application: hypothetical model for wood

For wood, and due to the different species, drying processes and modes of vibration, no collection of data is up to date available to permit a fair choice of the α and c parameters. Most of the experiments that were conducted on wood were rather limited to definite frequencies or frequency bands, either in the audio spectrum or in the ultrasonic range. Moreover, the largest part of the published material deals with investigations on the dynamic modulus, and very few publications were devoted to studying the loss factor.

The first difficulty in an attempt to present a model for wood stems from choosing an appropriate value of the relaxation time. For wood this value ought to be somewhere between the values taken by metals and those by polymeric materials. For many solid materials damping is caused by various mechanisms which are often intricate to formulate. However, heat conduction is often considered as one of the main causes of material damping, and this phenomenon has been studied so thoroughly that its interpretation may be considered as being well at hand. As far as wood is concerned, different damping mechanisms operate at

different frequency ranges. Other causes of damping have also been identified when dealing with materials, namely those due to plastic flow, as well as damping due to sound radiation or to the flow of vibrational energy in the system supporting the test sample. In what follows all these types of damping are discarded and only losses due to heat conduction are considered. Hence, incorporating heat conduction into the stress-strain relations leads to a relatively simple form of the loss factor due to heat conduction, namely, see for instance (Cremer and Heckl 1988, p. 235):

$$\eta(\omega) \propto \frac{\omega\tau_r}{1 + \omega^2\tau_r^2} \tag{17}$$

The frequency curve of this function is symmetrical on a log-log coordinate system. The curve has a low frequency asymptote increasing with 6 dB per octave (doubling of frequency), reaches a maximum at $\omega\tau_r = 1$, and then decreases by 6 dB/octave in the high frequency limit. At very high frequencies, that is for $\omega\tau_r >> 1$, the period of vibration is so short as compared to the relaxation time that heat can flow within the material permitting thus a temperature equalisation to be reached. The process is in this case said to be isothermal, and the loss factor becomes very small. In the other limiting case, for $\omega\tau_r$, too, the losses due to heat conduction are very small. In between these two extreme cases there exists a frequency region where the loss factor exhibits large values, the maximum of which is for $\omega\tau_r = 1$. For metals, and in the longitudinal mode of vibration, this happens at very high frequencies, around 10^{11} Hz, and the corresponding value for polymers lies often somewhere in the audio frequency range. Note here that the relaxation time depends strongly on the kind of excitation. For bending vibrations the regions of different temperatures are very close to each other. For a beam or a plate, the upper side, for instance, heated and compressed, and the lower side, extended and cooled, are at a distance equal to the thickness of the beam or the plate. It follows then that the frequency at which the maximum of damping occurs is very low, often at a value several orders of magnitude smaller than that for longitudinal motion, for test samples of usual sizes (Cremer and Heckl 1988). For bending vibrations, the value of the loss factor may then be higher and at the same time may rise more quickly as function of frequency than for the case of longitudinal vibrations (in the case of wood see for instance the figures in (Skudrzyk 1968, p. 111) where curves of the loss factor are plotted against frequency for different MC contents of wood).

Nonetheless, taking the plausible value of the frequency at which maximum longitudinal attenuation occurs in wood to be around 1.5 MHz, as taken from measurements conducted on the maximum attenuation of ultrasound in solid wood (Bucur and Böhnke 1994), then a value would be found for the relaxation time corresponding to approximately $\tau_r = 10^{-7}$ s. This value being shorter than that of polymers and longer than that of metals lies well within the range of the values expected.

The next step is to find appropriate values for α and c in the theory in the foregoing section. One method which possibly allows one to make a confident choice of these quantities would be to make some curve fitting on plots of experimental data. However, in lack of such data covering a broad enough range of frequencies, one can only rely on common sense to make acceptable choices. Comparison of the obtained frequency curves with available data permits then to validate or reject the model. Figure 1 shows the frequency curves of the dynamic modulus and the loss factor for $\alpha = 0.3$ and five different values of c ranging from $c = 1.5$ to $c = 3.5$. The static value of the Modulus of Elasticity was taken as $M_0 = 10^{10}$ Pa, a value typical for spruce. Considering the loss factor, one sees that the curve for $c = 1.5$ fits best to the experimental data available on wood, see for instance (Skudrzyk 1968, p. 487).

For higher values of c, the curves predict higher values of the loss factor, although the differences between the curves become less pronounced for increasing values of c. Similar curves are plotted in Fig. 2 for $c = 1.5$ and $0.1 < \alpha < 0.5$.

In this last set of curves a value of $\alpha = 0.3$ seems most appropriate since for this value the frequency dependence of the dynamic modulus is about what is found in literature. Lower values of α give too steep curves in the low frequencies range, and a too weak frequency dependence thereafter. In this case, the loss factor does not show appreciable variation within a broad frequency range; for $\alpha = 0.1$, for instance, the relative variation of the loss factor is only about 2% within 5 frequency decades. On the other hand, larger values of α predict too steep frequency curves at around the maximum loss factor after a too weak frequency dependence at lower frequencies. Hence, from both sets of figures, and at examining simultaneously the behavior of both the dynamic modulus and the loss factor, the values of $c = 1.5$ and $\alpha = 0.3$ may be considered as a satisfactory choice for modelling dispersion in wood. Note that the range of values $\alpha \leq 0.3$ is the range where excellent agreement is found between the exact theory, the Kramers-Kronig relations, and its approximation, the fractional Zener model (Pritz 1999).

A closer look at the final frequency curves in the audio frequency range is given in Fig. 3, where the exact curves for the dynamic modulus and the loss factor are compared to the corresponding approximate ones.

The curve of the dynamic modulus exhibits a frequency dependence that is not linear, and moreover, is relatively strong at low frequencies (note the log-scale on the frequency axis). The relative increment of the value of M_d is about 5% at 1.3 kHz and increases to around 10% at 20 kHz. The relative error committed when using the approximate expression for M_d, amounts to 5% only at a frequency as low as 100 Hz. The error is intolerable, 24% at 20 kHz. This is not so surprising since the approximate formulae are given under the assumption that $c >> 1$, whereas our value is only 1.5. For the loss factor, Fig. 3-right, things get even worse since the approximate formulae predict values that are on the average 2.5 times those given by the exact theory along almost the whole frequency range.

Conclusions

In this paper a review has been made of the causes of dispersion in wood. This phenomenon is found in all real solid materials and is a consequence of their non-

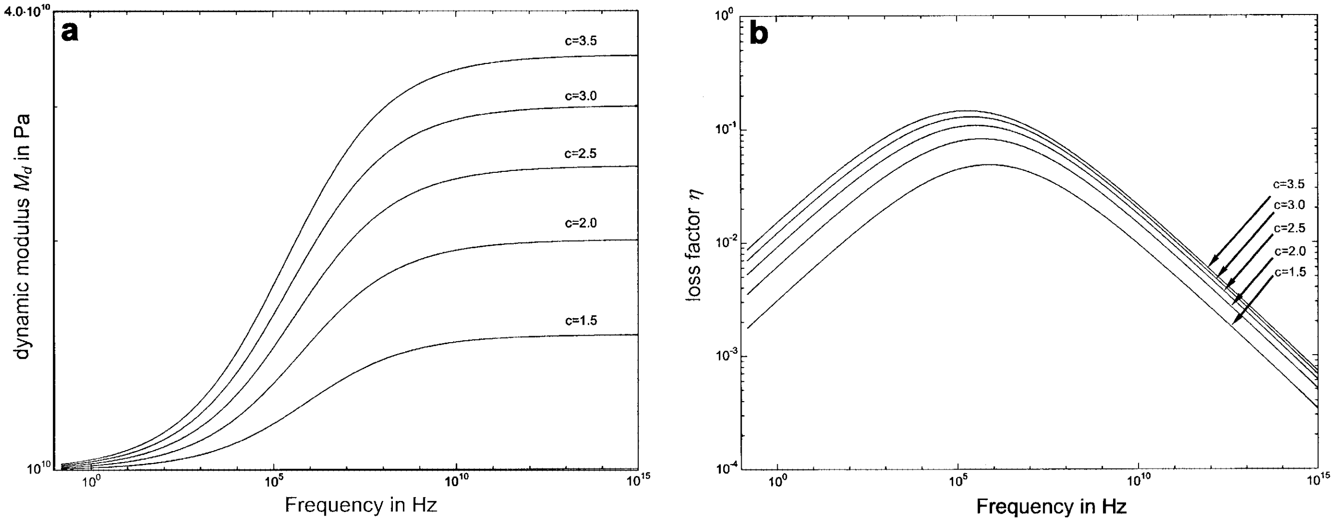

Fig. 1a,b. Hypothetical model for wood according to Eqs. (9) and (10) with $\alpha = 0.3$. **a** dynamic modulus, **b** loss factor

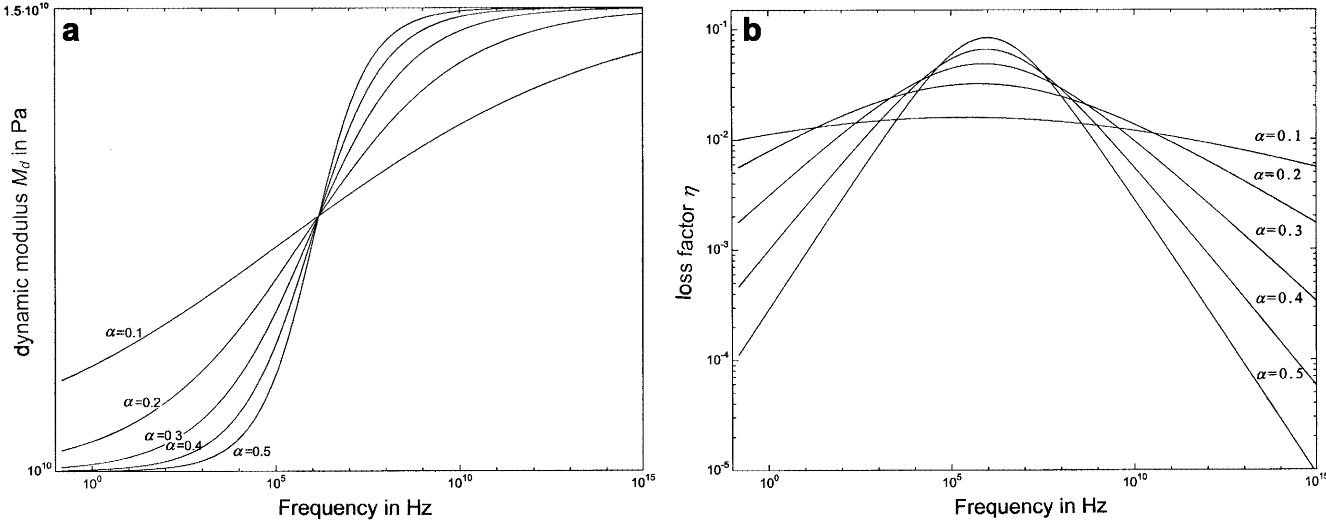

Fig. 2a,b. Hypothetical model for wood according to exact theory: and approximate for $c = 1.5$. **a** dynamic and loss moduli, **b** loss factor

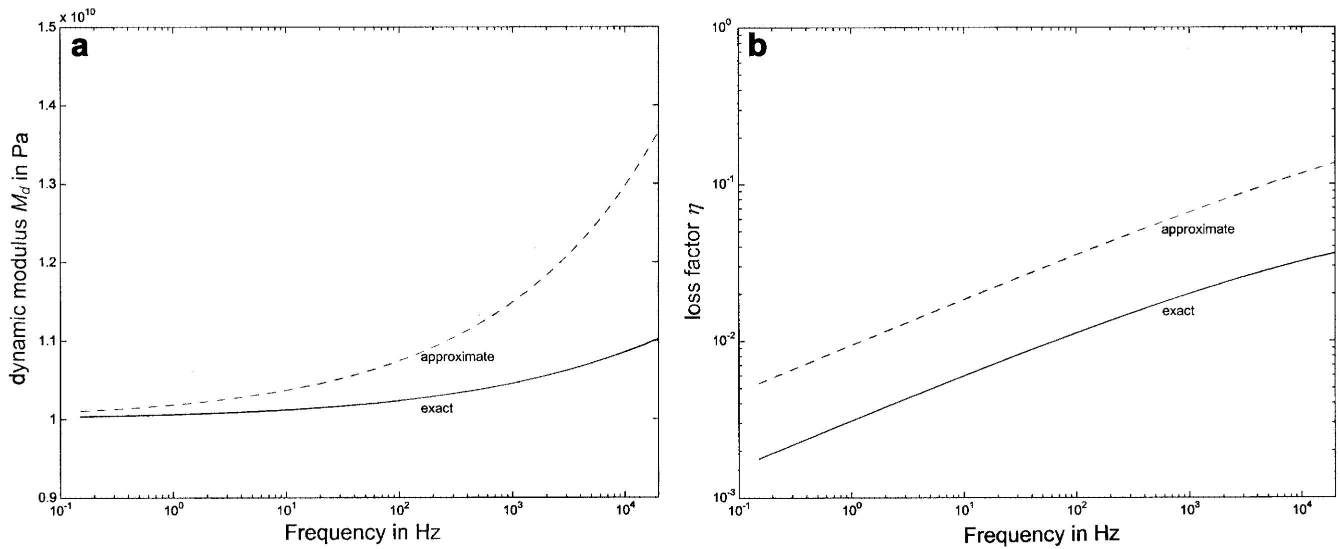

Fig. 3a,b. Frequency variation of the dynamic modulus, (**a**) and the loss factor, (**b**) for a hypothetical model for wood with $c = 1.5$ and $\alpha = 0.3$

ideal elasticity. An introduction has first been made to the necessary and useful concept of the complex Modulus of Elasticity, MOE, which is composed of a real component, the dynamic modulus, and an imaginary component, the loss modulus. The loss factor may then be calculated simply from taking the ratio of these two quantities. Dispersion finds its explanation in interpreting the exact formulation as given by the Kramers-Kronig relations when applied to the complex MOE. The exact formulae are, however, awkward to handle mathematically, and an approximation, namely the fractional Zener model, has been presented in this paper. This approximate model, having been successfully tested on viscoelastic materials, has also been applied to wood to build a model with fairly reasonable results. The main conclusion drawn from theory, is which also supported by experiments, is that the dynamic modulus is an increasing function of the frequency; the lowest possible value taken by the dynamic MOE being the static one.

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