Chapter 7: Kinetic Energy and Work

- Kinetic Energy
- Work (constant force)
- Work – Kinetic Energy Theorem
  - work done by a gravitational force
- Work (variable force)
  - work done by a spring force
- Power

Why do we study kinetic energy and work?

They provide us with a simpler way to calculate speed as a function of position.

Example 1

\[ F = 1.0 \, N \]

\[ v_i = 0 \quad \text{to} \quad v_f = ? \]

\[ 1 \text{ m} \]

A particle is moved 10 m by \( F = 1.0 \, N \). What is its final speed?
Answer:

\[ \text{work} = \text{change in kinetic energy} \]

\[ Fd \cos \phi = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \]

force

\[ (1)(10) \cos(0) = \frac{1}{2} \ 2 \ v_f^2 - \frac{1}{2} \ 2 \ v_i^2 \]

\[ 10 = v_f^2 \]

\[ \sqrt{10} \text{ m/s} = v_f \]

Example 2:

a particle slides along a track. what \( v_f \)?

\[ h = 10 \text{ m} \]

Solution:

\[ \text{work} = \text{change in kinetic energy} \]

\[ Fd \cos \phi = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \]

\[ mgh \cos \theta = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \]

\[ (9.8)(10) = \frac{1}{2} v_f^2 \]

\[ v_f = \sqrt{2(98)} \text{ m/s}. \]
**Kinetic Energy**

\[ K = \frac{1}{2} m v^2 \]

**Example:** a particle has a mass of 10 kg and a velocity of 20 m/s. What is its kinetic energy?

**Answer:**

\[ K = \frac{1}{2} m v^2 \]
\[ = \frac{1}{2} (10)(20)^2 \]
\[ = 2000 \text{ J} \]

In SI unit, energy is measured in Joule (J)

\[ 1 \text{ J} = 1 \text{ kg} \left( \frac{\text{m}}{\text{s}} \right)^2 \]

**Work**

\[ W = F \cdot d \]

Work done by a force on a particle.
Example:

\[ F = 10 \text{N} \]

\[ \theta = 45^\circ \]

\[ \vec{d} \]

\[ \vec{d} = 20 \text{m} \]

What is the work done on a particle by a constant force to move it 20 m?

Answer:

\[ W = F \cdot \vec{d} \]

\[ = (10)(20) \cos 45^\circ = \frac{200}{\sqrt{2}} \text{J} \]

Work is also measured in Joules

1 Joule = 1 N \cdot \text{m} = 1 \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{m}.

Example:

\[ F = 10 \text{N} \]

\[ \theta = 90^\circ \]

\[ \vec{d} \]

\[ \vec{d} = 20 \text{m} \]

Answer:

\[ W = F \cdot \vec{d} \]

\[ = Fd \cos 0 = 0 \]

Example:

\[ F = 10 \text{N} \]

\[ \theta = 135^\circ \]

\[ \vec{d} \]

\[ \vec{d} = 20 \text{m} \]

Answer:

\[ W = F \cdot \vec{d} \]

\[ = Fd \cos 135^\circ = -\frac{200}{\sqrt{2}} \text{J} \]
when the component of the force along the direction of \( \vec{d} \), work is positive

when \( \vec{F} \) is normal to \( \vec{d} \), \( W = 0 \)

when the component of the force is opposite to \( \vec{d} \), work is negative

**Work - Energy Theorem**

\[
\Delta K = W
\]

Change in Kinetic Energy

\[= K_f - K_i\]

Final Kinetic Energy \( \rightarrow \) Initial Kinetic Energy

Net work done on the particle.
\[ K_f - K_i = W \]
\[ K_f = K_i + W \]

Kinetic Energy after the work = Kinetic Energy before the work + the net work

If \( W \) is positive, final kinetic energy increases (velocity increases).
We say energy is transferred to the particle.

If \( W \) is negative, final kinetic energy decreases (velocity decreases).
We say energy is transferred from the particle.
Example: \( M = 225 \text{ kg} \), \( 10 \text{ N} \)

\[
\begin{align*}
\text{Frictionless} & \quad \quad 8.5 \text{ m} \\
= d & \\
\end{align*}
\]

Two forces move a block 8.5 m. What is the net work done on the block?

Answer:

Method 1

\[
W = W_1 + W_2 + W_g + W_N
\]

\[
\begin{align*}
\text{Work done by} & \quad \text{Work done by} \quad \text{Work done by} \quad \text{Work done by} \\
F_1 & \quad F_2 & \quad mg & \quad N
\end{align*}
\]

\[
= F_1 d \cos \phi_1 + F_2 d \cos \phi_2 + 0 + 0
\]

\[
= 10(8.5) \cos 40^\circ + 12(8.5) \cos 30^\circ
\]

\[
= 153 \text{ J}.
\]
method II: 

\[ W = \mathbf{F}_{\text{net}} \cdot \mathbf{d} = d \cdot \mathbf{F}_{\text{net}} = 0 \]

\[ = F_{\text{net},x} \, dx + F_{\text{net},y} \, dy \]

\[ = [10 \cos 40 + 12 \cos 30] \, (8.5) \]

\[ = 153 \, J. \]

What is the final speed

\[ W = K_f - K_i \]

\[ 153 \, J = \frac{1}{2} \, (255) \, u_f^2 - \frac{1}{2} \, 255 \, (0) \]

\[ u_f = \sqrt{\frac{2 \times 153}{255}} = 1.22 \, \text{m/s} \]

Example: An object slides a distance \( d \), and the following according to the work done by \( F \):

1. biggest work
2. zero work
3. least work

\[ F \]
Work done by a gravitational force

We know that the work done by a constant force is
\[ W = \mathbf{F} \cdot \mathbf{d} \]
\[ = Fd \cos \phi \]

A gravitational force \( \mathbf{F}_g \) is a constant force
\[ W_g = \mathbf{F}_g \cdot \mathbf{d} \]
\[ = mgd \cos \phi \]

---

### Rising particle
- Final \( W_g = mgd \cos 180^\circ \)
  - \( = -mgd \)
  - = negative work

### Falling particle
- Initial \( W_g = mgd \cos 90^\circ \)
  - \( = mgd \)
  - = positive work

---

Initial Energy transferred from the particle
\( \Rightarrow \) speed decreases

Final Energy transferred to the particle
\( \Rightarrow \) speed increases
Work done in lifting and lowering a particle

Suppose we use a vertical constant force to lift or lower a particle. The work-kinetic energy theorem tells us that the change in the kinetic energy of the particle is equal to the work done on the particle.

\[ \Delta K = K_f - K_i = W_g + W_F \]

If the initial and final speed of the particle are the same ⇒ \( K_f = K_i \)

\[ 0 = W_g + W_F \]

\[ W_F = -W_g \]

**Example**

\[ M = 15 \text{kg} \]

Frictionless surface

Initial stationary \( v_i = 0 \)

Final stationary \( v_f = 0 \)

\( h = 2.5 \text{ m} \)

\( 5 \text{ m} \)
Q. How much work $W_g$ is done on the block by the gravitational force?

A. $W_g = F_g \cdot \Delta \vec{d}$
   
   $= mg \Delta d \cos \phi$

To find $d \cos \phi$

**Method 1**

$\cos \phi = \cos (\theta + 90) = -\sin \theta$

$\Delta \sin \theta = h$

$\Rightarrow W_g = -mg \Delta h$

$= -(15)(9.8)(2.5) = -368 \text{ J}$.

**Method 2**

$W_g = F_g \cdot \Delta \vec{d}$

$\Delta \cdot \text{ dot product means the product of the magnitude of } F_g \text{ by the projection of } \Delta \text{ along } F_g$

(from figure $= -h$

$W_g = mg (-h)$

$= -368 \text{ J}$

Q. How much work $W_T$ is done by the tension $T$?

A. We do not know the value of $\vec{T}$, but we know $\Sigma T = -\Sigma i = 0$. Thus

$W_g + W_T + W_N = K_f - K_i = 0$

$\vec{L} = 0 \text{ since } \vec{N} \text{ is perpendicular to } \vec{d}$
\[ W_T = -W_g = -(-368) = 368 \text{ J} \]

Q. Compare tension and work \( W_T \) done by \( T \) for the following cases (Assume frictionless surface).

**CASE A**

\[ \dot{v}_i = \dot{v}_f = 0 \Rightarrow W_g + W_T + W_N^0 = \Delta K = 0 \]

\[ W_T = -W_g \]

In both cases, the projections of the displacement along \( \vec{F}_g \) are equal to \( h \).

\[ \Rightarrow W_{\text{gA}} = W_{\text{gB}} \]

\[ \Rightarrow W_{T_{\text{A}}} = W_{T_{\text{B}}} \]

The work done by \( T \) in both cases are equal.

\[ W_{T_{\text{A}}} = T_A d_A \]

\[ W_{T_{\text{B}}} = T_B d_B \]

\[ \begin{align*}
T_A d_A & = T_B d_B \\
\text{since } d_A & < d_B \\
\Rightarrow T_A & > T_B
\end{align*} \]
Example

Q. What is the work done by gravitational force?

A. \[ W_g = \vec{F}_g \cdot \vec{d} \]
\[ = mgd \cos \theta \]
\[ = (500)(9.8)(12) \]
\[ = 59 \text{ KJ} \]

Q. What is the work done by the tension?

Newton's second law:
\[ \vec{F}_{\text{net}} = ma \]
\[ T - mg = m(-a) \]
\[ T = m(g - a) \]
\[ = m(9 - \frac{g}{5}) = 500(9.8)(1 - \frac{1}{5}) \]
\[ = 500(9.8) \cdot \frac{4}{5} \]
\[ W_T = \vec{T} \cdot \vec{d} \]
\[ = Td \cos 180^\circ \]
\[ = (500)(9.8)(\frac{4}{5})(12)(-1) = -47 \text{ KJ} \]

Q. What is the final kinetic energy \( K_f \)?
Work - Kinetic Energy Theorem

\[ W_{\text{net}} = K_f - K_i \]
\[ W_g + W_T = K_f - K_i \]
\[ \frac{1}{2} m v_i^2 \]
\[ K_f = W_g + W_T + K_i \]
\[ = 59 \text{kJ} - 47 \text{kJ} + \frac{1}{2} (500)(4)^2 = 16 \text{kJ} \]

Work done by a variable force:

Suppose a particle moves from position \( \vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k} \) to \( \vec{r}_f = x_f \hat{i} + y_f \hat{j} + z_f \hat{k} \).

The work done by a variable force \( \vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \) is given by

\[ W = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz \]

For the variable force case, the work - kinetic energy theorem still holds: \( W = K_f - K_i \)

Spring force is an example of a variable force.
What is the spring force?

\[ \mathbf{F} = -k \mathbf{d} \]

Hooke's Law

Spring constant measured in \( \frac{N}{m} \).

One dimension:

\[ F = -k \times \]

Relaxed State

\[
\begin{align*}
F &= 0 \\
x &= 0
\end{align*}
\]

Compressed state

\[
\begin{align*}
F &= \text{positive} \\
x &= \text{negative}
\end{align*}
\]

Stretched State

\[
\begin{align*}
F &= \text{negative} \\
x &= \text{positive}
\end{align*}
\]
The work done by a spring force

\[ W_s = \frac{k}{2} \left( x_i^2 - x_f^2 \right) \]

\( x_i \) \hspace{1cm} \text{Initial position}

\( x_f \) \hspace{1cm} \text{Final position}

Q. Derive the above formula.

A. \[ W_s = \int_{x_i}^{x_f} F \, dx \]

\[ = \int_{x_i}^{x_f} (-kx) \, dx \]

\[ = -k \int_{x_i}^{x_f} x \, dx \]

\[ = -k \left[ \frac{x^2}{2} \right]_{x_i}^{x_f} \]

\[ = -\frac{k}{2} \left( x_f^2 - x_i^2 \right) \]

\[ = \frac{k}{2} \left( x_i^2 - x_f^2 \right). \]

Q. For the following situations, is the work done by the spring force positive, negative or zero?

\begin{align*}
\text{Initial} & \quad \begin{array}{c}
\text{Initial} \\
\hline
\text{Position} \\
\hline
\text{Final} \\
\hline
\text{Final} \\
\hline
\end{array} \\
\begin{array}{c}
\text{x=0} \\
\hline
\text{x=0} \\
\hline
\end{array} \\
\begin{array}{c}
\text{x=0} \\
\hline
\text{x=0} \\
\hline
\end{array} \end{align*}

\begin{align*}
\text{Final} & \quad \begin{array}{c}
\text{Final} \\
\hline
\text{Position} \\
\hline
\text{Initial} \\
\hline
\text{Initial} \\
\hline
\end{array} \\
\begin{array}{c}
\text{x=0} \\
\hline
\text{x=0} \\
\hline
\end{array} \\
\begin{array}{c}
\text{x=0} \\
\hline
\text{x=0} \\
\hline
\end{array} \end{align*}

\[ \begin{cases} \quad x_i^2 > x_f^2 \\ \quad \Rightarrow W_s = \text{positive} \end{cases} \]
When the final position is closer to the relaxed state than the initial position to the relaxed state, the $W_s$ is positive.

**Example:** A block is attached to a spring. An applied force of magnitude $F_a = 4.9 \text{ N}$ would be needed to hold the block at $x_i = 12 \text{ mm}$.
Q. What is the work \( W_s \) done by the spring force if the block moves from \( x_1 = 0 \) to \( x_2 = 17 \text{ mm} \)?

\[
A. \quad W_s = \frac{k}{2} (x_i^2 - x_f^2) = 0 \quad 17 \text{ mm}
\]

We do not know the spring constant \( k \) but we can find it from:

\[
F = -kx \Rightarrow k = -\frac{F}{x} = -\frac{(-4.9 \text{ N})}{12 \text{ mm}}
\]

\[
k = \frac{4.9}{12 \times 10^{-3}} \frac{\text{N}}{\text{m}}
\]

\[
W_s = \frac{1}{2} \frac{4.9}{12 \times 10^{-3}} (0 - (17 \times 10^{-3})^2) = -0.059 \text{ J}
\]

Example:

\[
k = \frac{750 \text{ N}}{\text{m}} \quad \nu = 0.15 \text{ m/s} \quad m = 0.4 \text{ kg}
\]

\[
u = 0 \text{ momentarily}
\]

\[
\Delta \nu = ?
\]

Oct 13, 01
What is the distance the spring compressed when the block is momentarily stopped?

\[ W_s = K_f - K_i \]

\[ \frac{k}{2} (x_i^2 - x_f^2) = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \]

\[ \frac{750}{2} (0 - d^2) = \frac{1}{2} (0.4)(0) - \frac{1}{2} (0.4)(0.5)^2 \]

\[ - \frac{750}{2} d^2 = - \frac{1}{2} (0.4)(0.5)^2 \]

\[ d = \sqrt{\frac{(0.4)(0.5)^2}{750}} = 1.2 \times 10^{-2} \text{ m} = 1.2 \text{ cm} \]

The work done by an applied force:

The work done by an applied force can be found from the work done by the spring force using:

\[ W_{\text{net}} = K_f - K_i \]

\[ W_s + W_a = K_f - K_i \]

For \( U_f = U_i \):

\[ W_a = -W_s \]
Power

\[ P_{avg} = \frac{W}{\Delta t} \]  

Average power

The average power \( P_{avg} \) due to a force during a time interval \( \Delta t \) is the work \( W \) done by this force during \( \Delta t \).

\[ P = \frac{\text{d}W}{\text{d}t} \]  

Power

The instantaneous power \( P \) due to a force is the time rate of doing work.

\[ P = F \cdot \overset{\rightarrow}{s} \]  

Work

\[ P_{avg} = \frac{W}{\Delta t} = \text{slope} \]

\[ P = \frac{\text{d}W}{\text{d}t} = \text{slope} \]  

Work vs. time

Time
<table>
<thead>
<tr>
<th></th>
<th>Power</th>
<th>Symbol</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>SI</td>
<td>Watt</td>
<td>W</td>
<td>J·s</td>
</tr>
<tr>
<td>English</td>
<td>horsepower</td>
<td>hp</td>
<td>550 ft·lb/s</td>
</tr>
</tbody>
</table>

Work can be measured in kilowatt-hour

1 Kilowatt-hour = 1 kW·h

\[ = 10^3\,W \cdot 3600\,s \]

\[ = 3.6 \times 10^6\,J \]

Q. Show that \( P = \vec{F} \cdot \vec{v} \)

A. Instantaneous Power

\[ = \frac{dW}{dt} \]

\[ = \frac{\vec{F} \cdot d\vec{r}}{dt} \]

\[ = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \]

Q. What is the power due to \( T \) on the block?

A. Since \( \vec{T} \) is perpendicular to \( \vec{v} \), \( P = 0 \)

\[ P = \vec{T} \cdot \vec{v} = T\,v \cos 90^\circ = 0 \]
Example

\[ \begin{align*}
F_1 &= 2.0 \text{ N} \\
\theta &= 60^\circ \\
\vec{u} &= 3.0 \text{ m/s}
\end{align*} \]

Q. What is the net power due to \( \vec{F}_1 \) and \( \vec{F}_2 \)?

A. 
\[ \begin{align*}
P_1 &= \vec{F}_1 \cdot \vec{u} \\
&= F_1 \, u \cos 180 \\
&= (2.0)(3.0)(-1) = -6.0 \text{ W}
\end{align*} \]

\[ \begin{align*}
P_2 &= \vec{F}_2 \cdot \vec{u} \\
&= F_2 \, u \cos 60^\circ \\
&= (4.0)(3.0)(\frac{1}{2}) = 6.0 \text{ W}
\end{align*} \]

\[ P_{\text{net}} = P_1 + P_2 = 0 \]

The net rate of transfer of energy to or from the block is zero. That is, the net work is not changing. Thus, the kinetic energy \( \frac{1}{2} mu^2 \) is not changing and so the speed of the block will remain 3.0 m/s.