Gravitation

- Newton's Law of gravitation
  - Two particles or spheres
  - More than two particles: superposition
  - Near Earth's surface
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- Gravitational potential energy
  - Two particles
  - More than two particles
  - Escape velocity

- Kepler's laws
  - Orbits
  - Areas
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- Satellites energy
Newton's Law of gravitation:

Every particle attracts any other particle with a gravitational force whose magnitude is given by:

\[ F = G \frac{M_1 M_2}{r^2} \]

- \( F \) is the gravitational force
- \( G \) is the gravitational constant \( = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \)
- \( M_1 \) and \( M_2 \) are the masses of the particles
- \( r \) is the distance between the two particles

\( F \) and \(-F\) form a third-law force pair; they are opposite in direction but equal in magnitude.
**Principle of superposition**

If you have more than two particles, the force on one particle is the vectorial sum of the forces between this particle and all other particles.

\[ \vec{F}_{\text{net},1} = \vec{F}_{12} + \vec{F}_{13} + \ldots \]

\[ \vec{F}_{14} = \text{force on particle one from particle four.} \]

**Example:** What is the force on \( m_1 \)?

\( m_1 = 6.0 \text{ Kg} \)
\( m_2 = 4.0 \text{ Kg} \)
\( m_3 = 4.0 \text{ Kg} \)
\( a = 2.0 \text{ cm} \)

\[ F_{12} = \frac{Gm_1m_2}{a^2} = 4.0 \times 10^{-6} \text{ N} \]
\[ F_{13} = \frac{Gm_1m_3}{(2a)^2} = 1.0 \times 10^{-6} \text{ N} \]
Since $F_{12}$ and $F_{13}$ are perpendicular,

$$F_{\text{net}} = \sqrt{(F_{12})^2 + (F_{13})^2} = 4.1 \times 10^{-6} \text{ N}$$

$$\phi = \tan^{-1} \left( \frac{F_{12}}{F_{13}} \right) = 76^\circ$$

$$\theta = 180^\circ - \phi = 104^\circ$$

When you want to find the gravitational force due to a uniform shell or sphere of mass $m$ on a particle outside the shell or sphere, you can treat the shell or sphere as a particle of mass $m$ located at the center of the shell or sphere.

$$m_1 \downarrow F = m_1 \downarrow F$$

$\text{shell or sphere}$
Newton's Shell Theorem

A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Example

Q: What is the force on \( m_1 \) and \( m_2 \) due to the shell?

A: \( F_{\text{shell}} = 0 \), because \( m_1 \) is inside the shell.

\[ F_{\text{shell}} = \frac{G m_2 m_3}{r^2} \]

We treat the shell as a particle located at the center.
Gravitation near Earth's surface:

When an object is close to Earth's surface, the gravitational force on the object due to Earth is approximately

\[ F = g \, m \]

This is true only for objects close to Earth's surface. Near Earth's surface, the distance between the object and the center of Earth is approximately the radius of Earth.

\[ F = \frac{GMm}{r^2} \]

\[ g \approx \frac{GM}{R^2} \]

\[ \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})}{(6.37 \times 10^6)^2} = 9.8 \, m/s^2 \]
Gravitation inside Earth

Assume Earth is a uniform sphere. To find the gravitational force on an object inside Earth, divide Earth into two parts. One part is the inner sphere on which the object lies. The other part is the outer shell.

\[
\text{object inside Earth } = m = \text{inner sphere } = M_{\text{sphere}}
\]

According to Newton's shell theorem, the shell exerts no net force on the object because the object is inside the shell. So the gravitational force is given by

\[
F = \frac{G M_{\text{sphere}} m}{r^2}
\]
Nov 24, 01

All Earth's mass sphere volume

\[
M_{\text{sphere}} = \frac{M_E}{V_E} V_{\text{sphere}}
\]

\[
= \frac{M_E}{\frac{4\pi R^3}{3}} \cdot \frac{4\pi r^3}{3} = \frac{M_E}{R^3} r^3
\]

\[
F = \frac{G\left(\frac{M_E}{R^3}\right) m}{r^2}
\]

\[
= \left[\frac{G M_E m}{R^3}\right] \frac{m}{r}
\]

This is a Spring Force (Hooke's law)

Gravitational Potential Energy:

Between two particles

\[
U = -\frac{G m_1 m_2}{r}
\]

gravitational potential energy
If one of the particle is a uniform shell or sphere, you can treat it as a particle located at the center of the shell or sphere.

Note we will not study the case when a particle is inside a shell or sphere.

Example The gravitational potential of a particle-Earth system is given by

\[ U = -\frac{G M_{E} m}{r} \]

1. \[ U = -\frac{G M_{E} m}{(R + h)} \]
   - Earth's radius
   - height from Earth's surface

Previously, we learned that the gravitational energy of a particle-Earth system is given by

\[ U = mgh \]

\[ U = 0 \text{ at } h = \infty \]
Q

What is the difference between Equation 1 and equation 2?

A

Both give the same potential energy difference for a particle near Earth’s Surface. Equation 2 is only valid near Earth’s surface and it is much easier to use. Equation 1 is true for any height. In definition 2 \( u = 0 \) at Earth’s surface while in definition 2 \( u = 0 \) at \( h = \infty \).

The quantity that affects physics is the difference in potential energy, not the absolute value of potential energy. Suppose a particle of mass \( m = 1 \text{kg} \) is moved from Earth’s surface to a height of \( h = 1 \text{Km} \).

\[ h = 1 \text{Km} \]

initial

final
Since \( h = 1 \text{ km} \) is much smaller than Earth's radius \( R = 6.37 \times 10^6 \text{ m} \), the particle is near Earth's surface and both equations should give same potential difference.

According to (2),

\[
\Delta U = U_f - U_i
\]

\[
= mg h - 0
\]

\[
= (1)(9.8)(1 \text{ K}) = 9.8 \text{ KJ}
\]

According to (1),

\[
\Delta U = U_f - U_i
\]

\[
= -G\frac{M_Em}{(R+h)} - (-G\frac{M_Em}{R})
\]

This is \( g = 9.8 \text{ m/s}^2 \)

\[
= G\frac{M_E}{R^2} m \left( -\frac{R^2}{R+h} + R \right)
\]

\[
= 9.8 \text{ KJ}
\]

\( G \approx h \)

\[
\text{Check } (-\frac{6.37 \times 10^6}{6.37 \times 10^6 + 10^3} + 1) \approx 999.8
\]

\[
\approx 1 \text{ K}
\]

\( \approx h \)
Gravitational potential energy for a system consisting of more than two particles.

\[ U = -G \left( \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right) \]

Example: what is the potential energy of this system?

\[ m_1 = m_2 = m_3 = 1 \text{ kg} \]
\[ r_{12} = r_{23} = 1 \text{ m} \]

\[ A_1 \quad U = -G \left[ \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right] \]
\[ = -6.67 \times 10^{-11} \left[ \frac{(1)(1)}{1} + \frac{(1)(1)}{\sqrt{2}(1)} + \frac{(1)(1)}{1} \right] \]
\[ = -6.67 \times 10^{-11} \left[ 2 + \frac{1}{\sqrt{2}} \right] \text{ J} \]
Q2 Suppose all particles at rest and an external force is used to move particle 3 to infinity. Assume the speed of particle 3 at infinity is zero. What is the work done by the external force on particle 3? 0 (no friction)

\[
W = \Delta K + \Delta U + \Delta E_{th}
\]

\[
= U_f - U_i
\]

\[
= (-G \frac{m_1 m_2}{r_{12}}) - (-G \left[ \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{13}} \right])
\]

\[
= 6.67 \times 10^{-11} \left( 1 + \frac{1}{\sqrt{2}} \right) J
\]

Q3 What is the work done by the gravitational force?

\[
W = -\Delta U = -6.67 \times 10^{-11} \left( 1 + \frac{1}{\sqrt{2}} \right) J
\]
Escape velocity :-

It is the speed of a particle on Earth's surface for which the particle will move to infinity. The speed of the particle at infinity is zero.

- $v=0$ at infinity

From conservation of energy

$K_i + U_i = K_f + U_f$

$\frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0 + 0$

$v = \sqrt{\frac{2GM}{R}} \approx 11.2 \text{ km/s}$
Kepler's Laws

- Kepler has derived law for planets' motions about the sun.
- All Kepler's laws can be derived from Newton's law of gravitation.
- Kepler's laws are applicable to moons or satellites orbiting a planet.
- Assumption: the mass of the orbiting object is much smaller than the mass of the fixed object.

The law of orbits: All planets move in elliptical orbits, with the sun at one focus.

What is an ellipse? An ellipse is described by two parameters.
1. semi-major axis \( a \)
2. eccentricity \( e \)

Equation of ellipse
\[
\frac{r_1}{a} + \frac{r_2}{a} = 2a
\]

Distance from focus 1
Distance from focus 2

Ellipse \( \rightarrow \) circle \( e \to 0 \)

The law of areas: A line that connects a planet to the sun sweeps out equal areas in equal times.

\[
\frac{dA}{dt} = \text{Constant}
\]
The law of periods: the square of the period of any planet is proportional to the cube of the semimajor axis of the orbit.

\[
\frac{T^2}{A^3} = \text{constant}
\]

\[
(\frac{4\pi^2}{GM})
\]

mass of the fixed object

Example

Halley's comet

\[ R_p = 8.9 \times 10^{10} \text{ m} \]
\[ M_s = 1.99 \times 10^{30} \text{ kg} \]
\[ T = 76 \text{ years} \]
Q. What is $R_p$?

A. \[ R_u + R_p = 2a \]

Given

\[ a = \left( \frac{GMT^2}{4\pi^2} \right)^{1/3} = 2.7 \times 10^7 \text{ m} \]

from law of periods

\[ R_p = 2(2.7 \times 10^7) - 8.9 \times 10^{10}. \]

Q. What is the eccentricity of the orbit of comet Halley?

A. \[ R_u + e a = a \]

\[ e = \frac{a - R_u}{a} = 0.97. \]

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**Satellite energy:**

Mechanical energy is conserved and it is given by

\[ E = -\frac{GMm}{2a} \]

does not depend on eccentricity.