1. Fill in the following table with the words TRUE or FALSE where appropriate.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>separable</th>
<th>exact</th>
<th>linear 1st order ode</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x^2 - y^2)dx + (e^y \ln y - 2xy)dy = 0)</td>
<td>FALSE</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>(xdy - (2y + x^3 \cos x)dx = 0)</td>
<td>FALSE</td>
<td>FALSE</td>
<td>TRUE</td>
</tr>
<tr>
<td>(\frac{dy}{dx} = a^{x+y} \quad (a &gt; 0, a \neq 1))</td>
<td>TRUE</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
</tbody>
</table>

2. Consider the initial value problem (IVP)

\[ y' = \sqrt{y^2 - 9} \]
\[ y(x_0) = y_0. \]

Describe \((x_0, y_0)\) for which the IVP will have unique solution

**Solution.** \(f(x, y) = \sqrt{y^2 - 9}\) is continuous in the region

\[ \{(x, y) : x \in \mathbb{R}, y^2 \geq 9\} = \{(x, y) : x \in \mathbb{R}, |y| \geq 3\} \]

But \(\frac{\partial y}{\partial x} = \frac{y}{\sqrt{y^2 - 9}}\) is continuous in the region

\[ \{(x, y) : x \in \mathbb{R}, y^2 > 9\} = \{(x, y) : x \in \mathbb{R}, |y| > 3\} \]

Therefore, the given IVP has unique solution through all points \((x_0, y_0)\) such that \(x_0 \in \mathbb{R}\) and \(|y_0| > 3\). This is the region

\[ \{(x, y) : x \in \mathbb{R}, y > 3\} \cup \{(x, y) : x \in \mathbb{R}, y < -3\} \]
3. Obtain a one parameter family of solutions to the differential equation

\[(1 + y^2)dx + \sqrt{1 - x^2}dy = 0\]

Your solution must be displayed in the explicit form.

**SOLUTION.** Separate the variables to get

\[
\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{1 + y^2} = 0
\]

Integrate to get

\[\sin^{-1} x + \tan^{-1} y = c\]

where \(c\) is an arbitrary constant. This solution can be expressed explicitly as

\[y(x) = \tan(c - \sin^{-1} x)\].

4. If

\[y = f(x; c_1, c_2, c_3)\]

where, \(c_1, c_2, c_3\) are arbitrary constants, is the solution of a linear differential equation, then the differential equation must be of order **THREE**. This solution is of the **EXPLICIT** form.

5. Let \(a\) be a non-zero constant. Solve

\[y + xy' = a(1 + xy)\].

**SOLUTION.** Open up the bracket to get \(y + xy' = a + axy\). Then collect together the terms in \(y\) to get \(xy' + (1 - ax)y = a\). This is then put in the standard form as

\[y' + \left(\frac{1}{x} - a\right)y = \frac{a}{x}\].

Therefore the integrating factor

\[
\mu = e^{\int \left(\frac{1}{x} - a\right)dx}
\]

\[= e^{\ln|x| - ax} = |x|e^{-ax} = |x|e^{-ax}\]
Choose \( \mu = xe^{-ax} \) and then multiply the standard form with \( \mu \) to get

\[
x e^{-ax} y' + x e^{-ax} \left( \frac{1}{x} - a \right) y = ae^{-ax}.
\]

That is, \((xe^{-ax}y)' = ae^{-ax}\) which, on integration, gives \(xe^{-ax}y = c - e^{-ax}\) where \(c\) is an arbitrary constant of integration. Therefore,

\[
y = \frac{c}{x} e^{ax} - \frac{1}{x}, \quad x \neq 0
\]

6. Obtain the critical points of the differential equation

\[
\frac{dy}{dx} = y \sin y
\]

Given that \(y = 0\) is a critical solution, determine whether it is stable, semi-stable or unstable

**SOLUTION** The critical points are given by \(c \sin c = 0\). This gives \(c = 0\) or \(\sin c = 0\). The latter yields \(c = n\pi, \ n = 0, \pm 1, \pm 2, \ldots\). Thus, the critical points are

\[c = 0, 0, \pm \pi, \pm 2\pi, \ldots\]

Now consider the equilibrium (critical) solution \(y_c(x) = 0\). When \(-\pi < y < 0\), then \(y' = y \sin y > 0\) and \(y\) is an increasing function. Similarly, when \(0 < y < \pi\), \(y' = y \sin y > 0\) and and again \(y\) is an increasing function. Thus \(y_c(x) = 0\) is an attractor for \(y < 0\) and a repeller for \(y > 0\). Therefore, \(y_c(x) = 0\) is *semi-stable.*

\[
\begin{array}{c}
\pi \\
+ \\
+ \\
+ \\
0 \\
+ \\
+ \\
-\pi \\
Phase Portrait Around \(c = 0\)
\end{array}
\]