NONLINEAR VARIATIONAL INEQUALITIES FOR PSEUDOMONOTONE OPERATORS WITH APPLICATIONS∗

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Abstract. In this paper, we prove the existence of solutions to the variational and variational-like inequalities for pseudomonotone and pseudodissipative and, η-pseudomonotone and η-pseudodissipative operators, respectively. As applications of our results, we prove the existence of a unique solution of nonlinear equations, fixed point problems and eigenvalue problems.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a real locally convex Hausdorff topological vector space with topological dual X∗ and K a non-empty subset of X. Let T : K → X∗ be an operator and η : K × K → X a bifunction. The variational-like inequality problem (for short, VLIP) is to find x̄ ∈ K such that

⟨T(x̄), η(y, x̄)⟩ ≥ 0, for all y ∈ K,

where ⟨u, x⟩ denotes the pairing between u ∈ X∗ and x ∈ X. For further details on VLIP, we refer to [2, 5, 9-12, 16] and references therein.

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When \( \eta(y, x) = y - x \), the VLIP reduces to the variational inequality problem (for short, VIP) [7] of finding \( \bar{x} \in K \) such that

\[
\langle T(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all} \ y \in K.
\]

In most of the results on the existence of solutions to the VIP and VLIP some kind of continuity assumption on the operator \( T \) is needed if it has some kind of monotonicity assumption, see for example [3-4, 6-8, 12-15, 17-18] and references therein.

The main object of this paper is to establish some existence results for VIP and VLIP in the setting of non-compact convex set \( K \) with pseudomonotone and pseudodissipative and, \( \eta \)-pseudomonotone and \( \eta \)-pseudodissipative operator \( T \), respectively. As applications of our results, we prove the existence of a unique solution of nonlinear equations, fixed point problems and eigenvalue problems without any continuity assumption on the operator \( T \).

We shall use the following notation and definitions. Let \( A \) be a non-empty set. We shall denote by \( 2^A \) the family of all subsets of \( A \). If \( A \) and \( B \) are non-empty subsets of a topological vector space \( Y \) such that \( A \subseteq B \), we shall denote by \( \text{int}_B A \) the interior of \( A \) in \( B \).

The inverse \( F^{-1} \) of a multivalued map \( F : X \to 2^Y \) is the multivalued map from \( \text{R}(F) \), the range of \( F \), to \( X \) defined by

\[
x \in F^{-1}(y) \quad \text{if and only if} \quad y \in F(x).
\]

We shall use the following particular form of Corollary 1 in [1].

**Lemma 1.1.** Let \( K \) be a non-empty and convex subset of a Hausdorff topological vector space \( E \), and let \( S : K \to 2^K \) be a multivalued map. Assume that the following conditions hold.

(a) For each \( x \in K \), \( S(x) \) is non-empty and convex.

(b) \( K = \bigcup \{ \text{int}_K S^{-1}(y) : y \in K \} \).

(c) If \( K \) is not compact, assume that there exists a non-empty, compact and convex subset \( C \) of \( K \) and a non-empty and compact subset \( D \) of \( K \) such that for each \( x \in K \setminus D \), there exists \( \tilde{y} \in C \) such that \( x \in \text{int}_K S^{-1}(\tilde{y}) \).

Then \( S \) has a fixed point, that is, there exists \( x_0 \in K \) such that \( x_0 \in S(x_0) \).

**2. EXISTENCE RESULTS**

For a given bifunction \( \eta : K \times K \to X \), an operator \( T : K \to X^* \) is called:
(i) \(\eta\)-monotone if,
\[
\langle T(y) - T(x), \eta(y, x) \rangle \geq 0, \quad \text{for all } x, y \in K;
\]

(ii) \(\eta\)-dissipative if,
\[
\langle T(y) - T(x), \eta(y, x) \rangle \leq 0, \quad \text{for all } x, y \in K;
\]

(iii) \(\eta\)-pseudomonotone if,
\[
\langle T(x), \eta(y, x) \rangle \geq 0 \quad \text{implies} \quad \langle T(y), \eta(y, x) \rangle \geq 0, \quad \text{for all } x, y \in K,
\]

or equivalently,
\[
\langle T(y), \eta(y, x) \rangle < 0 \quad \text{implies} \quad \langle T(x), \eta(y, x) \rangle < 0, \quad \text{for all } x, y \in K;
\]

(iv) \(\eta\)-pseudodissipative if,
\[
\langle T(y), \eta(y, x) \rangle \geq 0 \quad \text{implies} \quad \langle T(x), \eta(y, x) \rangle \geq 0, \quad \text{for all } x, y \in K,
\]

or equivalently,
\[
\langle T(x), \eta(y, x) \rangle < 0 \quad \text{implies} \quad \langle T(y), \eta(y, x) \rangle < 0, \quad \text{for all } x, y \in K.
\]

When \(\eta(y, x) = y - x\), the definitions of \(\eta\)-monotone, \(\eta\)-dissipative, \(\eta\)-pseudomonotone and \(\eta\)-pseudodissipative reduce to the definitions of monotone, dissipative [17], pseudomonotone and pseudodissipative, respectively.

\textbf{Example 2.1.} Let \(T : \mathbb{R} \to \mathbb{R}\) be defined as
\[
T(x) = \begin{cases} 
1 & : x \neq 1 \\
2 & : x = 1
\end{cases}
\]

Then \(T\) is pseudomonotone as well as pseudodissipative but it is neither monotone nor hemicontinuous.

For \(\eta(y, x) = y^2 - x^2\), \(T\) is also \(\eta\)-pseudomonotone as well as \(\eta\)-pseudodissipative but not \(\eta\)-monotone.

An example of a pseudomonotone hemicontinuous operator is given in [15] which is not continuous on finite dimensional spaces.

\textbf{Theorem 2.1.} Let \(K\) be a non-empty and convex subset of a locally convex Hausdorff topological vector space \(X\) and let \(\eta : K \times K \to X\) be a bifunction such that \(\eta(x, x) = 0\), for all \(x \in K\). Assume that
(i) \( T : K \to X^* \) is \( \eta \)-pseudomonotone and \( \eta \)-pseudodissipative;

(ii) for each fixed \( y \in K \), the map \( x \mapsto \langle T(y), \eta(y, x) \rangle \) is upper semicontinuous on \( K \);

(iii) for each fixed \( x \in K \), the map \( y \mapsto \langle T(x), \eta(y, x) \rangle \) is quasi-convex;

(iv) there exists a non-empty, compact and convex subset \( C \) of \( K \) and a non-empty and compact subset \( D \) of \( K \) such that for each \( x \in K \setminus D \), there exists \( \tilde{y} \in C \) such that \( \langle T(x), \eta(\tilde{y}, x) \rangle < 0 \).

Then the VLIP has a solution.

**Proof.** Assume that the VLIP has no solution. Then for each \( x \in K \),

\[ \{ y \in K : \langle T(x), \eta(y, x) \rangle < 0 \} \neq \emptyset. \]

We define a multivalued map \( S : K \to 2^K \) by

\[ S(x) = \{ y \in K : \langle T(x), \eta(y, x) \rangle < 0 \}, \quad \text{for all } x \in K. \]

Then clearly for all \( x \in K \), \( S(x) \neq \emptyset \). From assumption (iii), it is easy to see that \( S(x) \) is convex, for all \( x \in K \). Now

\[ S^{-1}(y) = \{ x \in K : \langle T(x), \eta(y, x) \rangle < 0 \}. \]

For each \( y \in K \), we denote by \( [S^{-1}(y)]^c \) the complement of \( S^{-1}(y) \) in \( K \). From the \( \eta \)-pseudomonotonicity of \( T \), we have

\[ [S^{-1}(y)]^c = \{ x \in K : \langle T(x), \eta(y, x) \rangle \geq 0 \} \]
\[ \subseteq \{ x \in K : \langle T(y), \eta(y, x) \rangle \geq 0 \} \]
\[ = H(y) \text{(say)}. \]

From condition (ii), it is easy to show that for all \( y \in K \), \( H(y) \) is closed in \( K \).

From the \( \eta \)-pseudodissipativeness of \( T \), we have

\[ S^{-1}(y) = \{ x \in K : \langle T(x), \eta(y, x) \rangle < 0 \} \]
\[ \subseteq \{ x \in K : \langle T(y), \eta(y, x) \rangle < 0 \} \]
\[ = [H(y)]^c, \text{ the complement of } H(y) \text{ in } K. \]

Hence \( S^{-1}(y) = [H(y)]^c \) and \( S^{-1}(y) \) is open in \( K \). Since \( S(x) \neq \emptyset \), we have

\[ K = \bigcup_{y \in K} S^{-1}(y) = \bigcup_{y \in K} \text{int}_K S^{-1}(y). \]
By assumption (iv), for each \(x \in K \setminus D\), there exists \(\tilde{y} \in C\) such that \(\langle T(x), \eta(\tilde{y}, x) \rangle < 0\), we have \(x \in \text{int}_K S^{-1}(\tilde{y})\). Then \(S\) satisfies all the conditions of Lemma 1.1, hence there exists \(x_0 \in K\) such that \(x_0 \in S(x_0)\), that is,

\[\langle T(x_0), \eta(x_0, x_0) \rangle < 0.\]

Since \(\eta(x_0, x_0) = 0\), we have

\[0 = \langle T(x_0), \eta(x_0, x_0) \rangle < 0,\]

a contradiction. Hence the result is proved. 

REMARK 2.1. If \(X\) is a reflexive Banach space equipped with weak topology, then the assumption (iv) in Theorem 2.1 can be replaced by the following condition:

(iv)' There exists \(\tilde{y} \in K\) such that \(\liminf_{\|x\| \to \infty, x \in K} \langle T(x), \eta(\tilde{y}, x) \rangle < 0\).

PROOF. By (iv)', there exists \(r > 0\) such that \(\|\tilde{y}\| < r\) and if \(x \in K\) with \(\|x\| \geq r\), we have \(\langle T(x), \eta(\tilde{y}, x) \rangle < 0\). Define \(B_r = \{x \in K : \|x\| \leq r\}\). Then \(B_r\) is a non-empty weakly compact and convex subset of \(X\). By taking \(C = D = B_r\) in assumption (iv) of Theorem 2.1, we get the conclusion. 

In view of Remark 2.1, we have the following result.

COROLLARY 2.1. Let \(K\) be a non-empty and convex subset of a reflexive Banach space \(X\) equipped with weak topology and let \(\eta : K \times K \to X\) be a bifunction such that it is affine in the first argument, weakly continuous in the second argument and \(\eta(x, x) = 0\), for all \(x \in K\). Assume that \(T : K \to X^*\) is \(\eta\)-pseudomonotone, \(\eta\)-pseudodissipative and there exists \(\tilde{y} \in K\) such that \(\liminf_{\|x\| \to \infty, x \in K} \langle T(x), \eta(\tilde{y}, x) \rangle < 0\). Then the VLIP has a solution.

COROLLARY 2.2. Let \(K\) be a non-empty and convex subset of a locally convex Hausdorff topological vector space \(X\) and let \(T : K \to X^*\) be pseudomonotone and pseudodissipative. Assume that there exists a non-empty, compact and convex subset \(C\) of \(K\) and a non-empty and compact subset \(D\) of \(K\) such that for each \(x \in K \setminus D\), there exists \(\tilde{y} \in C\) such that \(\langle T(x), \tilde{y} - x \rangle < 0\). Then the VIP has a solution.

REMARK 2.2. In the results of Browder [3-4], Hartman and Stampacchia [6] (Theorem 1.1), Tarafdar [13] (Theorem 2 and Corollary), Verma [14] (Theorem 2.2) and Yao [18] (Theorem 3.3), we need continuity/hemicontinuity/continuity on finite dimensional spaces. But in Corol-
lary 2.2 we do not assume any kind of continuity assumption.

**Corollary 2.3.** Let $K$ be a non-empty and convex subset of a reflexive Banach space $X$ equipped with weak topology and let $T : K \rightarrow X^*$ be pseudomonotone, pseudodissipative and has the property that there exists $\tilde{y} \in K$ such that $\lim \inf_{\|x\| \rightarrow \infty, x \in K} \langle T(x), \tilde{y} - x \rangle < 0$. Then the VIP has a solution. Moreover, if $T$ is strongly pseudomonotone then the solution is unique.

**Remark 2.3.** Corollary 2.3 is different from Theorems 3.1 and 3.2 in [17] in the following ways:

(a) $X$ need not be a Hilbert space,
(b) $K$ need not be closed,
(c) $T$ need not be continuous on finite-dimensional subspaces,
(d) $T$ need not be hemicontinuous,
(e) $T$ is assumed only pseudomonotone and pseudodissipative, need not be monotone.

**3. APPLICATIONS**

Throughout this section, we will assume that $H$ is a real Hilbert space with its inner product denoted by $(.,.)$.

Let $K$ be a non-empty subset of $H$. An operator $T : K \rightarrow K$ is called:

(i) **strongly monotone** if, there exists a constant $\alpha > 0$ such that $$(T(y) - T(x), y - x) \geq \alpha \|y - x\|^2,$$ for all $x, y \in K$;

(ii) **relaxed strongly monotone** if, there exists a constant $\beta < 1$ such that $$(T(y) - T(x), y - x) \leq \beta \|y - x\|^2,$$ for all $x, y \in K$;

(iii) **relaxed strongly dissipative** if, there exists a constant $\nu < 1$ such that $$(T(y) - T(x), y - x) \geq \nu \|y - x\|^2,$$ for all $x, y \in K$;

(iv) **strongly pseudomonotone** if, there exists a constant $\gamma > 0$ such that $$(T(x), y - x) \geq 0 \text{ implies } (T(y), y - x) \geq \gamma \|y - x\|^2,$$ for all $x, y \in K$. 


We now give the following result concerning the existence of a unique solution of a nonlinear equation.

**Theorem 3.1.** Let $T : H \to H$ be pseudomonotone, pseudodissipative and assume that there exists $\tilde{y} \in H$ such that $\lim \inf_{||x|| \to \infty} (T(x), \tilde{y} - x) < 0$. Then there exists $\bar{x} \in H$ such that $T(\bar{x}) = 0$. Moreover, if $T$ is strongly pseudomonotone then the solution is unique.

**Proof.** It is similar to the proof of Theorem 3.3 in [17].

**Remark 3.1.** Theorem 3.1 is different from Theorem 3.3 in [17] in the following ways:

(a) $T$ need not be hemicontinuous,

(b) $T$ is assumed only pseudomonotone and pseudodissipative, need not be monotone.

By using the results of Section 2, we establish the following fixed point theorem.

**Theorem 3.2.** Let $K$ be a non-empty and convex subset of $H$ and $T : K \to K$ be relaxed strongly monotone and relaxed strongly dissipative. Then $T$ has a unique fixed point.

**Proof.** It is similar to the proof of Theorem 3.4 in [17].

**Remark 3.2.** Theorem 3.2 is different from Theorem 3.4 in [17] in the following ways:

(a) $K$ need not be closed,

(b) $T$ is assumed relaxed strongly dissipative, need not be hemicontinuous.

Finally, we derive the following existence results for solutions to the eigenvalue problem.

**Corollary 3.1.** Let $K$ be a non-empty convex cone of $H$ and $T : K \to K$ be monotone and dissipative. Then for any nonnegative real number $\lambda$ and any $z \in K$, there exists a unique $\bar{x} \in K$ such that $\lambda T(\bar{x}) + z = \bar{x}$.

**Proof.** It is similar to the proof of Corollary 3.7 in [17].

**Remark 3.3.** Corollary 3.1 is different from Corollary 3.7 in [17] in the following ways:

(a) $K$ need not be closed,
(b) $T$ is assumed monotone, need not be hemicontinuous.

REFERENCES


