LOCAL EXISTENCE AND BLOW UP IN NONLINEAR THERMOELASTICITY WITH SECOND SOUND

Salim A. Messaoudi

Mathematical Sciences Department, King Fahd University of Petroleum and Minerals (KFUPM), Dhahran 31261, Saudi Arabia
E-mail: messaoud@kfupm.edu.sa

ABSTRACT

In this work we establish a local existence and a blow up result for a multidimensional nonlinear system of thermoelasticity with second sound.

Key Words: Thermoelasticity; Second sound; Nonlinear source; Negative initial energy; Local existence; Blow up; Finite time

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1. INTRODUCTION

Results concerning existence, blow up, and asymptotic behaviors of smooth, as well as weak, solutions in classical thermoelasticity have been
established by several authors over the past two decades. See in this regard
Refs. [1–3,5,6,8–15,18,20].
For thermoelasticity with second sound, global existence of smooth
solutions for the one-dimensional case, has been established by Tarabek.[21]
In his work, the author used the usual energy argument to prove his result.
Saouli[18] used the nonlinear semigroup theory presented by Kato[4] to prove
a local existence result for a system similar to the one considered in Ref. [21].
Concerning the asymptotic behavior, Racke[16] discussed lately the
one-dimensional situation and established exponential decay results for sev-
eral initial boundary value problems. In particular he showed that, for small
enough initial data, classical solutions of a certain nonlinear problem decay
exponentially to the equilibrium state. Regarding the multi-dimensional
case \(n = 2, 3\) Racke[17] established an existence result for the following
\(n\)-dimensional problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \mu \Delta u - (\mu + \lambda) \nabla \cdot \nabla u + \beta \nabla \theta &= 0 \\
\theta_t + \gamma \nabla q + \delta \nabla u_t &= 0 \\
\tau q_t + q + \kappa \nabla \theta &= 0, \quad x \in \Omega, t > 0 \\
u(.,0) &= u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\
u = \theta = 0, \quad x \in \partial \Omega, t \geq 0,
\end{align*}
\] (1.1)
where \(\Omega\) is a bounded domain of \(\mathbb{R}^n\), with a smooth boundary \(\partial \Omega\), \(u = u(x, t) \in \mathbb{R}^n\) is the displacement vector, \(\theta = \theta(x, t)\) is the difference temperature, \(q = q(x, t) \in \mathbb{R}^n\) is the heat flux vector, and \(\mu, \lambda, \beta, \gamma, \delta, \tau, \kappa\) are positive constants, where \(\mu, \lambda\) are Lame moduli and \(\tau\) is the relaxation time, a small parameter compared to the others. In particular if \(\tau = 0\), (1.1) reduces to the system of classical thermoelasticity, in which the heat flux is given by Fourier’s law instead of Cattaneo’s law. He also proved, under the conditions \(\text{rot} \ u = \text{rot} \ q = 0\), an exponential decay result for (1.1). This result is extended
to the radially semmetric solution, as it is only a special case.

In this paper we are concerned with the nonlinear problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \mu \Delta u - (\mu + \lambda) \nabla \cdot \nabla u + \beta \nabla \theta &= |u|^{p-2}u \\
\theta_t + \gamma \nabla q + \delta \nabla u_t &= 0 \\
\tau q_t + q + \kappa \nabla \theta &= 0, \quad x \in \Omega, \quad t > 0 \\
u(.,0) &= u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\
u = \theta = 0, \quad x \in \partial \Omega, \quad t \geq 0,
\end{align*}
\] (1.2)
for \(p > 2\). This is a similar problem to (1.1) with a nonlinear source term
competing with the damping factor. We will establish a local existence
result and show that solutions with negative energy blow up in finite time.
NONLINEAR THERMOELASTICITY

This work generalizes the one in Refs. [8,9] to thermoelasticity with second sound. This paper is organized as follows: in section two we establish the local existence. In section three the blow up result is proved.

2. LOCAL EXISTENCE

In this section, we establish a local existence result for (1.2) under a suitable condition on \( p \). First we establish an existence result for a related linear problem

\[
\begin{align*}
    u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla \cdot u + \beta \nabla \theta &= f \\
    \theta_t + \gamma \nabla q + \delta \nabla u_t &= 0 \\
    r_t + q + \kappa \nabla \theta &= 0, \quad x \in \Omega, \quad t > 0 \\
    u(.,0) &= u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\
    u &= \theta = 0, \quad x \in \partial \Omega, t \geq 0.
\end{align*}
\]

(2.1)

For this purpose we introduce the following spaces

\[
\begin{align*}
    \Pi &:= [H^1_0(\Omega) \cap H^2(\Omega)]^n \times [H^1_0(\Omega)]^n \times H^1_0(\Omega) \times D \\
    D &:= \{ q \in [L^2(\Omega)]^n / \text{div } q \in L^2(\Omega) \} \\
    H &:= [H^1_0(\Omega)]^n \times [L^2(\Omega)]^n \times L^2(\Omega) \times [L^2(\Omega)]^n \\
    \Lambda &:= \max_{0 \leq t \leq T} \left\{ \| (u, u_t, \theta, q)(., t) \|^2_{H^1_0} + \| (u_t, u_{tt}, \theta, q_t)(., t) \|^2_H \right\} \\
    \Lambda_0 &:= \| (u_0, u_1, \theta_0, q_0) \|^2_{H^1_0} + \| (u_t, u_{tt}, \theta_1, q_1) \|^2_H,
\end{align*}
\]

where

\[
\begin{align*}
    u_t &= \mu \Delta u_0 + (\mu + \lambda) \nabla \cdot u_0 - \beta \nabla \theta_0 + f (., 0) \\
    \theta_1 &= -\gamma \nabla q_0 - \delta \nabla u_1 \\
    q_1 &= -[q_0 + \kappa \nabla \theta_0] / \tau.
\end{align*}
\]

Lemma 2.1. Assume that \( f \in C^1([0, T); L^2(\Omega))^n \). Then given any initial data \((u_0, u_1, \theta_0, q_0) \in \Pi\), the problem (2.1) has a unique strong solution satisfying

\[
(u, u_t, \theta, q) \in C^1([0, T); \Pi) \cap C([0, T); H).
\]

Moreover we have

\[
\Lambda \leq \Gamma \Lambda_0 + \Gamma T \max_{0 \leq t \leq T} \left\{ \| f(., t) \|^2_{H^1_0} + \| f_t(., t) \|^2_H \right\},
\]

(2.7)

where \( \Gamma \) is a constant depending on \( \mu, \lambda, \beta, \gamma, \delta, \kappa, \tau \) only.
Proof. The existence of solutions satisfying (2.6) is a direct result of Theorem 2.2 of Ref. [17]. To establish (2.7), we multiply (2.1) by $u_i$, $\beta \theta/\delta$, $\beta \gamma q/(\delta \kappa)$ respectively and integrate over $\Omega \times (0, t)$ to get
\[
\frac{1}{2} \int_{\Omega} \left[ |u|^2 + \mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + \frac{\beta}{\delta} |\theta|^2 + \frac{\gamma \beta r}{\delta \kappa} |q|^2 \right] (x, t) \, dx
\]
\[
= \frac{1}{2} \int_{\Omega} \left[ |u|^2 + \mu |\nabla u_0|^2 + (\lambda + \mu) (\text{div} u_0)^2 + \frac{\beta}{\delta} |\theta_0|^2 + \frac{\gamma \beta r}{\delta \kappa} |q_0|^2 \right] (x) \, dx
\]
\[
+ \int_0^t \int_{\Omega} f(x, s) u_i(x, s) \, dx \, ds. \tag{2.8}
\]
To obtain estimates on terms involving higher order derivatives, we apply the difference operator
\[
\Delta_h w(x, t) := w(x + h, t) - w(x, t), \quad x \in \Omega, \quad t \in [0, T), \quad 0 < h < T - t
\]
to the Eq. (2.1). By multiplying the resulting equations by $\Delta_h u_i$, $\beta \Delta_h \theta/\delta$, $\beta \gamma \Delta_h q/(\delta \kappa)$ respectively, integrating over $\Omega \times (0, t)$, using integration by parts, dividing by $h^2$, and letting $h$ go to zero we arrive at
\[
\frac{1}{2} \int_{\Omega} \left[ |u|^2 + \mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + \frac{\beta}{\delta} |\theta|^2 + \frac{\gamma \beta r}{\delta \kappa} |q|^2 \right] (x, t) \, dx
\]
\[
= \frac{1}{2} \int_{\Omega} \left[ |u_1|^2 + \mu |\nabla u_1|^2 + (\lambda + \mu) (\text{div} u_1)^2 + \frac{\beta}{\delta} |\theta_1|^2 + \frac{\gamma \beta r}{\delta \kappa} |q_1|^2 \right] (x) \, dx
\]
\[
+ \int_0^t \int_{\Omega} f_j(x, s) u_{ij}(x, s) \, dx \, ds. \tag{2.9}
\]
By combining (2.8), (2.9), the (2.1), and using Cauchy-Schwarz inequality, (2.7) is established.

Lemma 2.2. Assume that
\[
2 < p \leq \frac{2(n - 3)}{n - 4}, \quad n \geq 5 \tag{2.10}
\]
and $v \in C([0, T]; H^2(\Omega))^n \cap (C^1([0, T]; H^1(\Omega))^n$. Then $f = |v|^{p-2} v$ satisfies
\[
\int_{\Omega} |f(x, t)|^2 \, dx \leq C \|v||H^2_p|^2, \quad \int_{\Omega} |f_j(x, t)|^2 \, dx \leq C \|v||H^1_p||v||L^p|^2, \tag{2.11}
\]
where $C$ is a constant depending on $\Omega$ and $p$ only.

The proof is trivial. We only use the embedding of Sobolev spaces in the $L^p$ spaces.

Remark 2.1. For $n \leq 4$, (2.11) remains valid without imposing (2.10).
NONLINEAR THERMOELASTICITY 1685

Theorem 2.3. Assume that (2.10) holds. Then given any \((u_0, u_1, \theta_0, q_0) \in \Pi\), the problem (1.2) has a unique strong solution satisfying (2.6), for \(T\) small enough.

Proof. For \(M > 0\) large and \(T > 0\), we define a class of functions \(Z(M, T)\) which consists of all functions \((w, \phi, \xi)\) satisfying (2.6), the initial conditions of (1.2), and

\[
\max_{0 \leq t \leq T} \left\{ \|(u, u_1, \theta, q)(., t)\|_{H^1}^2 + \|(u, u_1, \theta, q)(., t)\|_{H^1}^2 \right\} \leq M^2. \tag{2.12}
\]

\(Z(M, T)\) is nonempty if \(M\) is large enough. This follows from the trace theorem.\(^7\) We also define the map \(F\) by \((u, \theta, q) := F(w, \phi, \xi)\), where \((u, \theta, q)\) is the unique solution of the linear problem

\[
\begin{align*}
  u_t - \mu \Delta u - (\mu + \lambda) \nabla \div u + \beta \nabla \theta &= |v|^{p-2} v, \\
  \theta_t + \gamma \div q + \delta \div u_t &= 0, \\
  \tau q_t + q + \kappa \nabla \theta &= 0, \quad x \in \Omega, \quad t > 0 \\
  u(., 0) &= u_0, \quad u_t(., 0) = u_1, \quad \theta(., 0) = \theta_0, \quad q(., 0) = q_0, \quad x \in \Omega \\
  u &= \theta = 0, \quad x \in \partial \Omega, \quad t \geq 0
\end{align*}
\]

(2.13)

since \(|v|^{p-2} v \in [L^2(\Omega)]^n\) by virtue of (2.10). We would like to show, for \(M\) sufficiently large and \(T\) sufficiently small, that \(F\) is a contraction from \(Z(M, T)\) into itself.

By using (2.7) and (2.11) we get

\[
\max_{0 \leq t \leq T} \left\{ \|(u, u_1, \theta, q)(., t)\|_{H^1}^2 + \|(u, u_1, \theta, q)(., t)\|_{H^1}^2 \right\} \leq \Gamma \Lambda_0 + \Gamma CT \max_{0 \leq t \leq T} \left\{ \|v\|_{H^2}^{2p-2} + \|v_t\|_{H^1}^2 \|v_t\|_{H^1}^{2p-4} \right\} \leq \Gamma \Lambda_0 + \Gamma CT M^{2p-2}.
\]

By choosing \(M\) large enough and \(T\) sufficiently small, (2.12) is established; hence \((u, \theta, q) \in Z(M, T)\). So \(F\) maps \(Z(M, T)\) into itself.

Next we prove that \(F\) is a contraction. For this aim we equip \(Z(M, T)\) with the complete\(^1\) metric

\[
d((v^m, \phi^m, \xi^m), (v^l, \phi^l, \xi^l))
\]

\[
= \sqrt{\max_{0 \leq t \leq T} \left\{ \|(v^m - v^l, \phi^m - \phi^l, \theta^m - \theta^l, q^m - q^l)(., t)\|_{H^1}^2 \right\}}
\]

\(^1\)The completeness of the metric \(d\) follows from the weak * precompactness of bounded sets in \(L^\infty([0, T]: L^2(\Omega))\) and sequential weak lower semicontinuity of norms in these spaces (see Ref. [20]).
and set
\[ U := u^m - u^l, \quad \Theta := \theta^m - \theta^l, \quad Q = q^m - q^l \]
\[ V := v^m - v^l, \quad \Phi := \phi^m - \phi^l, \quad \xi = \xi^m - \xi^l \]
where \((u^m, \theta^m, q^m) = F(v^m, \phi^m, \xi^m)\) and \((u^l, \theta^l, q^l) = F(v^l, \phi^l, \xi^l)\). It is straightforward to see that \((U, \Theta, Q)\) satisfies
\[
U_t - \mu \Delta U - (\mu + \lambda) \nabla \cdot U + \beta \nabla \Theta = |v^m|^{p-2}v^m - |v^l|^{p-2}v^l \\
\Theta_t + \gamma \nabla Q + \delta \nabla U = 0 \\
\tau Q_t + Q + \kappa \nabla \Theta = 0, \quad x \in \Omega, \quad t > 0 \\
U(., 0) = U(., 0) = \Theta(., 0) = Q(., 0) = 0, \quad x \in \Omega \\
U = \Theta = 0, \quad x \in \partial \Omega, \quad t \geq 0.
\]
We multiply (2.14) by \(U_t, \beta \Theta / \delta, \beta \gamma Q / (\delta \kappa)\) respectively and integrate over \(\Omega \times (0, t)\) to get
\[
\frac{1}{2} \int_{\Omega} [\|U_t\|^2 + |\nabla U|^2 + (\lambda + \mu)(\nabla \cdot U)^2 + \frac{\beta}{\delta} |\Theta|^2 + \frac{\gamma \beta \tau}{\delta \kappa} |Q|^2] \, dx \\
\leq \int_0^t \int_{\Omega} \|v^m|^{p-2}v^m - |v^l|^{p-2}v^l\| U_t(x, s) \, dx \, ds \\
\leq C \int_0^t \|U_t\| \|V\|_{H^2} \left\{ \|v^m\|_{H^2}^{p-2} + \|v^l\|_{H^2}^{p-2} \right\} (., s) \, ds.
\]
Therefore (2.15) yields
\[
d((u^m, \theta^m, q^m), (u^l, \theta^l, q^l)) \leq \Gamma TM^{p-2} d((v^m, \phi^m, \xi^m), (v^l, \phi^l, \xi^l)).
\]
By choosing \(T\) so small that \(\Gamma TM^{p-2} < 1\), the estimate (2.16) shows that \(F\) is a contraction. The contraction mapping theorem then guarantees the existence of a unique \((u, \theta, q)\) satisfying \((u, \theta, q) = F(u, \theta, q)\). Obviously it is the unique solution of (1.2). The proof is completed.

3. BLOW UP RESULT

In this section we show that the solution (2.6) blows up in finite time if \(E(0) < 0\), where
\[
E(t) := -\frac{1}{p} \int_{\Omega} |u(x, t)|^p \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 + \mu |\nabla u|^2 \, dx \\
+ \frac{1}{2} \int_{\Omega} \left( (\mu + \lambda)(\nabla u)^2 + \frac{\beta}{\delta} \theta^2 + \frac{\gamma \beta \tau}{\delta \kappa} q^2 \right) \, dx.
\]
Lemma 3.1. Suppose that
\[ 2 < p \leq \frac{2n}{n-2}, \quad n \geq 3. \]  
(3.2)

Then there exists a positive constant \( C > 1 \) depending on \( \Omega \) and \( p \) only such that
\[ \|u\|_p^s \leq C\left(\|\nabla u\|_2^2 + \|u\|_p^p\right) \]  
(3.3)

for any \( u \in H_0^1(\Omega) \) and \( 2 \leq s \leq p \).

Proof. If \( \|u\|_p \leq 1 \) then \( \|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2 \) by Sobolev embedding theorems. If \( \|u\|_p > 1 \) then \( \|u\|_p^s \leq \|u\|_p^p \). Therefore (3.3) follows.

We set
\[ H(t) := -E(t) \]  
(3.4)

and use, throughout this paper, \( C \) to denote a generic positive constant depending on \( \Omega \) and \( p \) only. As a result of (3.1)–(3.4), we have

Corollary 3.2. Assume that (3.2) holds. Then
\[ \|u\|_p^s \leq C \left\{ \left(1 + \frac{2}{p\mu}\right)\|u\|_p^p - \frac{2}{\mu}\frac{\partial}{\partial t}H(t) - \frac{1}{\mu}\|u_t\|_2^2 \right\} + \left(1 + \frac{\lambda}{\mu}\right)\|\text{div} u\|_2^2 - \frac{\beta}{\delta\mu}\|\theta\|_2^2 - \frac{\gamma\beta\tau}{\delta\kappa}\|q\|_2^2 \right\}, \]  
(3.5)

for any \( u \in (H_0^1(\Omega))^n \) and \( 2 \leq s \leq p \).

Theorem 3.3. Assume that (3.2) holds. Assume further that
\[ p(p + 2) > \frac{\beta\tau\delta}{\kappa\gamma}. \]  
(3.6)

Then for any initial data in \( \Pi \) satisfying
\[ E(0) < 0, \]  
(3.7)

the solution (2.6) blows up in finite time.

Remark 3.1. The condition (3.6) is ‘physically’ reasonable due to the very small value of \( \tau \). For instance in Ref. [16], for the isotropic silicon and a medium temperature of 300K we have
\[ \beta \approx 391.62 \left[ \frac{m^2}{s^2 K} \right], \quad \tau \approx 10^{-12}[s], \quad \delta \approx 163.82[K], \]  
\[ \gamma \approx 5.99 \times 10^{-7} \left[ \frac{ms^2 K}{kg} \right], \quad \kappa \approx 148 \left[ \frac{W}{mK} \right]. \]
consequently we get
\[
\frac{\beta \tau \delta}{\kappa Y} \approx 72.367 \times 10^{-7}
\]

So (3.6) is satisfied for any \( p > 2 \).

**Remark 3.2.** If \( \tau = 0 \), then (1.2) reduces to the classical system of thermoelasticity and the blow up result takes place without condition (3.6). This is exactly what was proven in Refs. [8,9]. See also remarks by the end of Ref. [10].

**Proof.** We multiply Eq. (1.2) by \(-u_t, -\beta \theta /\delta, -\beta \gamma q /\delta\tau\) respectively and integrate over \( \Omega \), using integration by parts, and add equalities to get
\[
H'(t) = \frac{\gamma \beta}{\delta \kappa} \|q\|_2^2 \geq 0, \quad \forall t \in [0, T);
\]
consequently we get
\[
0 < H(0) \leq H(t), \quad \forall t \in [0, T),
\]
by virtue of (3.1) and (3.4). We then introduce
\[
L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega \left[ u.u_t + \frac{\beta \tau}{\kappa} u.q \right](x, t) \, dx
\]
for \( \varepsilon \) small to be chosen later and
\[
\alpha = \frac{(p-2)}{(2p)}.
\]
By taking a derivative of (3.10) and using Eq. (1.2) we obtain
\[
L'(t) = (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left[ \|u\|_{p}^p + \|u_t\|_{2}^2 - \mu \|\nabla u\|_{2}^2 - (\mu + \lambda) \|\nabla u\|_{2}^2 \right] \\
- \frac{\beta}{\kappa} \int_\Omega u.q \, dx + \varepsilon \frac{\beta \tau}{\kappa} \int_\Omega u_t.q \, dx.
\]
We then use (3.1) and (3.4) to substitute for \( \|u\|_{p}^p \); hence (3.12) takes the form
\[
L'(t) = (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left( \frac{P}{2} + 1 \right) \|u_t\|_{2}^2 + \mu \varepsilon \left( \frac{P}{2} - 1 \right) \|\nabla u\|_{2}^2 \\
+ \varepsilon (\mu + \lambda) \left( \frac{P}{2} - 1 \right) \|\nabla u\|_{2}^2 + \varepsilon \frac{\beta \gamma \tau}{2\delta} \|q\|_{2}^2 \\
+ \varepsilon \rho H(t) - \frac{\beta \tau}{\kappa} \int_\Omega u.q \, dx + \varepsilon \frac{\beta \tau}{\kappa} \int_\Omega u_t.q \, dx.
\]
We then exploit Young’s inequality to estimate the last two terms in (3.13) as follows

\[ \left| \int_{\Omega} u_1 q \, dx \right| \leq \frac{a}{2} \| u_1 \|_2^2 + \frac{1}{2a} \| q \|_2^2, \quad \forall a > 0 \]

\[ \int_{\Omega} u q \, dx \leq \frac{b}{2} \| u \|_2^2 + \frac{1}{2b} \| q \|_2^2, \quad \forall b > 0. \]

Thus (3.13) yields

\[
L'(t) \geq (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left( \frac{p + 2}{2} - \frac{a \beta \tau}{2 \kappa} \right) \| u_1 \|_2^2 + \mu \varepsilon \left( \frac{p}{2} - 1 \right) \| \nabla u \|_2^2 \\
+ \varepsilon (\mu + \lambda) \left( \frac{p}{2} - 1 \right) \| \text{div } u \|_2^2 + \varepsilon \frac{p \beta}{2 \delta} \| \theta \|_2^2 + \varepsilon \frac{\beta \tau}{2 \kappa} \left( \frac{\rho'}{\delta} - \frac{1}{a} \right) \| q \|_2^2 \\
+ \varepsilon p H(t) - \varepsilon \frac{\beta \tau}{\kappa} \left( \frac{b}{2} \| q \|_2^2 + \frac{1}{2b} \| u \|_2^2 \right). \tag{3.14}
\]

At this point we choose \( a \) so that

\[
A_1 := \frac{p + 2}{2} - \frac{a \beta \tau}{2 \kappa} > 0, \quad A_2 := \frac{\beta \tau}{2 \kappa} \left( \frac{\rho'}{\delta} - \frac{1}{a} \right) > 0.
\]

This is possible by virtue of (3.6); hence (3.14) becomes

\[
L'(t) \geq (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon A_1 \| u_1 \|_2^2 + \varepsilon A_2 \| q \|_2^2 + \varepsilon A_3 \| \nabla u \|_2^2 \\
+ \varepsilon A_4 \| \text{div } u \|_2^2 + \varepsilon A_5 \| \theta \|_2^2 + \varepsilon p H(t) - \varepsilon \frac{\beta \tau}{\kappa} \left( \frac{b}{2} \| q \|_2^2 + \frac{1}{2b} \| u \|_2^2 \right), \tag{3.15}
\]

where \( A_1-A_5 \) are strictly positive constants depending only on \( p, \beta, \gamma, \delta, \kappa, \lambda, \mu, \tau \). We also set \( b = 2 M \gamma H^{-\alpha}(t) / \delta \); for \( M \) a constant to be determined; hence (3.15) gives

\[
L'(t) \geq [(1 - \alpha) - \varepsilon M] H^{-\alpha}(t) H'(t) + \varepsilon A_1 \| u_1 \|_2^2 + \varepsilon A_2 \| q \|_2^2 + \varepsilon A_3 \| \nabla u \|_2^2 \\
+ \varepsilon A_4 \| \text{div } u \|_2^2 + \varepsilon A_5 \| \theta \|_2^2 + \varepsilon p H(t) - \varepsilon \frac{C \varepsilon}{4M} H^p(t) \| u \|_p^p, \tag{3.16}
\]

where \( C, \) here and in the sequel, is a positive generic constant depending on \( \Omega, p, \beta, \gamma, \delta, \kappa, \lambda, \mu, \tau \) only. We then use \( H(t) \leq \| u \|_p^p / p \) to get, from (3.16),

\[
L'(t) \geq [(1 - \alpha) - \varepsilon M] H^{-\alpha}(t) H'(t) + \varepsilon A_1 \| u_1 \|_2^2 + \varepsilon A_2 \| q \|_2^2 + \varepsilon A_3 \| \nabla u \|_2^2 \\
+ \varepsilon A_4 \| \text{div } u \|_2^2 + \varepsilon A_5 \| \theta \|_2^2 + \varepsilon p H(t) - \varepsilon \frac{C \varepsilon}{4M} \left( \frac{1}{p} \right) \| u \|_p^{2+\alpha p}. \tag{3.17}
\]
Since $2 + \alpha p < p$ we exploit (3.5) to obtain, from (3.17),

$$L'(t) \geq [(1 - \alpha) - \varepsilon M]H^{-\alpha}(t)H'(t) + \varepsilon \left( A_1 + \frac{C}{M} \right) \|u_t\|_{L^2}^2$$

$$+ \varepsilon \left( A_2 + \frac{C}{M} \right) \|q\|_2^2 + \varepsilon A_3 \|\nabla u\|_{L^2}^2 + \varepsilon \left( A_4 + \frac{C}{M} \right) \|	ext{div } u\|_{L^2}^2$$

$$+ \varepsilon \left( A_5 + \frac{C}{M} \right) \|\theta\|_2^2 + \varepsilon \left( p + \frac{C}{M} \right) H(t) - \frac{C\varepsilon}{M} \left( 1 + \frac{2}{p\mu} \right) \| u \|_{H^1}^p.$$  

(3.18)

At this point, we choose $M$ large enough so that the coefficients of the terms in (3.18) are strictly positive; hence we get

$$L'(t) \geq [(1 - \alpha) - \varepsilon M]H^{-\alpha}(t)H'(t)$$

$$+ \varepsilon A_0 \left[ H(t) + \|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|	ext{div } u\|_{L^2}^2 + \|q\|_{L^2}^2 + \|u\|_{H^1}^p \right].$$  

(3.19)

where $A_0 > 0$ is the minimum of these coefficients. Once $M$ is fixed (hence $A_0$), we pick $\varepsilon$ small enough so that $(1 - \alpha) - \varepsilon M \geq 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0(u_1 + \frac{\beta t}{\kappa} q)(x) \, dx > 0.$$

Therefore (3.19) leads to

$$L'(t) \geq A_0 \varepsilon \left[ H(t) + \|u_t\|_{L^2}^2 + \|q\|_{L^2}^2 + \|u\|_{H^1}^p \right].$$  

(3.20)

Consequently we have

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0.$$

Next we estimate

$$\left| \int_\Omega uu_t(x, t) \, dx \right| \leq C\|u\|_{L^2}^2 \|u_t\|_{L^2} \leq C\|u\|_{H^1} \|u_t\|_{L^2},$$

which implies

$$\left| \int_\Omega uu_t(x, t) \, dx \right|^{1/(1-\alpha)} \leq C\|u\|_{H^1}^{1/(1-\alpha)} \|u_t\|_{L^2}^{1/(1-\alpha)}.$$

Again Young’s inequality gives us

$$\left| \int_\Omega uu_t(x, t) \, dx \right|^{1/(1-\alpha)} \leq C \left[ \|u\|_{H^1}^{r/(1-\alpha)} + \|u_t\|_{L^2}^{r/(1-\alpha)} \right],$$  

(3.21)

for $1/r + 1/s = 1$. We take $s = 2(1 - \alpha)$, to get $r/(1 - \alpha) = 2/(1 - 2\alpha) = p$ by virtue of (3.11). Therefore (3.21) becomes

$$\left| \int_\Omega uu_t(x, t) \, dx \right|^{1/(1-\alpha)} \leq C \left[ \|u\|_{H^1}^p + \|u_t\|_{L^2}^2 \right], \quad \forall t \geq 0.$$  

(3.22)
Similarly we have
\[
\left| \int_{\Omega} uq(x, t) \, dx \right|^{1/(1-\alpha)} \leq C \left[ \|u\|_{p}^{2} + \|u_t\|_{2}^{2} \right], \quad \forall t \geq 0.
\]
(3.23)

Finally by noting that
\[
L^{1/(1-\alpha)}(t) = \left( H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u \left( u_t + \frac{\beta}{k} q \right)(x, t) \, dx \right)^{1/(1-\alpha)}
\]
\[
\leq C \left( H(t) + \left\| uu_t(x, t) \right\|_{(1-\alpha)}^{1/(1-\alpha)} + \left\| uq(x, t) \right\|_{1/(1-\alpha)}^{1/(1-\alpha)} \right)
\]
\[
\leq C \left[ H(t) + \|u\|_{p}^{2} + \|u_t\|_{2}^{2} + \|q\|_{2}^{2} \right], \quad \forall t \geq 0.
\]

and combining it with (3.20), (3.22), (3.23) we obtain
\[
L'(t) \geq a_0 L^{1/(1-\alpha)}(t), \quad \forall t \geq 0
\]
(3.24)

where \( a_0 \) is a positive constant depending on \( \varepsilon A_0 \) and \( C \). A simple integration of (3.24) over \((0, t)\) then yields
\[
L^{(p-2)/(p+2)}(t) \geq \frac{1}{L^{-(p-2)/2}(0) - a_0 t (p-2)/2}.
\]

Therefore \( L(t) \) blows up in a time
\[
T^* \leq \frac{1 - \alpha}{a_0 [L(0)]^{(p-2)/(p+2)}},
\]
(3.25)

Remark 3.3. The estimate (3.25) shows that the larger \( L(0) \) is the quicker the blow up takes place.

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