On the control of solutions of
a viscoelastic equation

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1 Introduction

Cavalcanti et al. studied

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + a(x)u_t \\
  + |u|^{\gamma}u = 0, \; \gamma > 0, \; \text{in } \Omega \times (0, \infty) \\
  u(x, t) = 0, \; x \in \partial \Omega, \; t \geq 0 \\
  u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \; x \in \Omega,
\end{cases}
\]  

(1)

\( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) bounded with \( \partial \Omega \) regular,
\( g \geq 0 \) with

\[-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \; t \geq 0\]

and \( ||g||_{L^1((0, \infty))} \) is small enough,
\( a : \Omega \to \mathbb{R}^+ \) such that

\[a(x) \geq a_0 > 0 \; \text{on } \phi \neq \omega \subset \Omega,\]

with \( \omega \) satisfying some geometry restrictions.

- An exponential decay result obtained.
- This extends the result of Zuazua, with \( g = 0 \).
- Berrimi and Messaoudi obtained the same result under weaker conditions on \( g \) and \( \omega \).

- Cavalcanti et al. considered

\[u_{tt} - k_0 \Delta u + \int_0^t \text{div}[a(x) g(t - \tau) \nabla u(\tau)] d\tau + b(x) h(u_t) + f(u) = 0, \; \text{in } \Omega \times (0, \infty),\]

under similar conditions on \( g \) and

\[a(x) + b(x) \geq \delta > 0, \forall \; x \in \Omega.\]
They established an exponential stability for \(g\) decaying exponentially and \(h\) linear and polynomial stability for \(g\) decaying polynomially and \(h\) nonlinear.

- Cavalcanti et al have also studied

\[
|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t - \tau)\Delta u(\tau)d\tau \\
- \gamma \Delta u_t = 0, \quad \rho > 0, \quad x \in \Omega, \quad t > 0,
\]

A global existence result for \(\gamma \geq 0\) and an exponential decay for \(\gamma > 0\) were established.

- This last result has been extended to a situation, where a source term is present, by Messaoudi and Tatar.

- Also, Messaoudi considered

\[
u_{tt} - \Delta u + \int_0^t g(t - \tau)\Delta u(\tau)d\tau \\
+ au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0, \infty)
\]

and showed, under suitable conditions on \(g\), that solutions with negative energy blow up in finite time if \(\gamma > m\) and continue to exist if \(m \geq \gamma\).

In the absence of the viscoelastic term \((g = 0)\), the problem has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the problem

\[
\begin{cases}
u_{tt} - \Delta u + au_t|u_t|^m = b|u|^\gamma u, & \text{in } \Omega \times (0, \infty) \\
u(x, t) = 0, & x \in \partial\Omega, t \geq 0 \\
u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega,
\end{cases}
\]

\(a, b, m, \gamma \geq 0\)

- For \(a = 0\), the source term \(bu|u|^{\gamma}\) \((\gamma > 0)\) causes finite time blow up of solutions with negative initial energy.

- For \(b = 0\), the damping term \(au_t|u_t|^m\) assures global existence for arbitrary initial data.

- For \(ab \neq 0\), in the linear damping case \((m = 0)\), a blow up result established by Levine for solutions with negative initial energy.

- Georgiev and Todorova studied the nonlinear damping case \((m > 0)\). They showed that solutions with negative energy continue to exist globally 'in time' if \(m \geq \gamma\) and blow up in finite time if \(\gamma > m\) and the initial energy is sufficiently negative.

- Messaoudi proved the blow up result for solutions with negative initial energy only.

- Results of same nature were established by Levine and Serrin, and Levine and Park.
Vitillaro, Messaoudi and Said-Houari, and others.

Our problem

\[
\begin{align*}
\begin{cases}
    u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau = 0, & \text{in } \Omega \times (0, \infty) \\
    u(x, t) = 0, & x \in \partial \Omega, t \geq 0 \\
    u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega,
\end{cases}
\end{align*}
\]  \hspace{1cm} (3)

Conditions

(G1) \(g : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is a bounded \(C^1\) function such that

\[
g(0) > 0, \quad 1 - \int_0^\infty g(s) ds = l > 0.
\]

(G2) There exists a positive constant \(\xi\) such that

\[
g'(t) \leq -\xi g(t), \quad t \geq 0.
\]

Proposition Let \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\) be given. Assume that \(g\) satisfies (G1). Then problem (3) has a unique global solution

\[
\begin{align*}
    u & \in C \left([0, \infty); H^1_0(\Omega)\right), \\
    u_t & \in C \left([0, \infty); L^2(\Omega)\right).
\end{align*}
\] \hspace{1cm} (4)

Remark Condition (G1) is necessary to guarantee the hyperbolicity of the system (3).

The modified energy

\[
\mathcal{E}(t) := \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u(t)\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t),
\] \hspace{1cm} (5)

where

\[
(g \circ v)(t) = \int_0^t g(t - \tau) \|v(t) - v(\tau)\|_2^2 d\tau.
\] \hspace{1cm} (6)

Lemma The modified energy satisfies

\[
\mathcal{E}'(t) = \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 \leq \frac{1}{2} (g' \circ \nabla u)(t) \leq 0.
\] \hspace{1cm} (7)
2 Exponential decay

**Theorem** Let \((u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)\) be given. Assume that \(g\) satisfies \((G1)\) and \((G2)\). Then there exist positive constants \(k\) and \(K\) such that the solution given by (2.2) satisfies

\[ E(t) \leq K e^{-kt}, \quad \forall t \geq t_0 > 0. \]

**Proof** We define

\[ F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t) \]

where \(\varepsilon_1\) and \(\varepsilon_2\) are positive constants to be specified later and

\[ \Psi(t) := \int_{\Omega} uu_t dx \]

\[ \chi(t) := -\int_{\Omega} u_t \int_0^t g(t - \tau)(u(t) - u(\tau))d\tau dx. \]

It is straightforward to see that for \(\varepsilon_1\) and \(\varepsilon_2\) so small, we have

\[ \alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t), \]

holds for two positive constants \(\alpha_1\) and \(\alpha_2\).

Direct, but lengthy, estimates give

\[ \Psi'(t) \leq \int_{\Omega} u_t^2 dx - \frac{l}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{(1 - l)}{2l} (g \circ \nabla u)(t). \]

Similarly

\[ \chi'(t) \leq \delta \{1 + 2(1 - l)^2 \|\nabla u\|_2^2 \]

\[ + (2\delta + \frac{1}{2\delta})(1 - l)(g \circ \nabla u)(t) \]

\[ + \frac{g(0)}{4\delta} C_p (-(g' \circ \nabla u)(t)) \]

\[ + (\delta - \int_0^t g(s) ds) \int_{\Omega} u_t^2 dx. \]

Since \(g(0) > 0\) then there exists \(t_0 > 0\) such that

\[ \int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0 > 0, \quad \forall t \geq t_0. \]
A combination of (10) - (12) leads to

\[ F'(t) \leq -\varepsilon_2\{g_0 - \delta\} - \varepsilon_1 \int_{\Omega} u_t^2 \, dx \]
\[ - \left[ \frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta \{1 + 2(1 - l)^2\}\right] \|\nabla u\|_2^2 \]
\[ + \left[ \frac{1}{2} - \frac{\varepsilon_1 (1 - l)}{2 \xi l} - \varepsilon_2 \frac{g(0)}{4 \delta} C_p \right] \\
+ \frac{(1 - l)}{\xi} \left( \frac{l}{2} + \frac{1}{2 \delta} \right) \right] (g' \circ \nabla u)(t). \]

(13)

Choose \( \delta \) so small that

\[ g_0 - \delta > \frac{1}{2} g_0 \]
\[ \frac{1}{l} \delta \{1 + 2(1 - l)^2\} < \frac{1}{8} g_0. \]

Whence \( \delta \) is fixed, the choice of any two positive constants \( \varepsilon_1 \) and \( \varepsilon_2 \) satisfying

\[ \frac{1}{4} g_0 \varepsilon_2 < \varepsilon_1 < \frac{1}{2} g_0 \varepsilon_2 \]

(14)

will make

\[ k_1 = \varepsilon_2\{g_0 - \delta\} - \varepsilon_1 > 0 \]
\[ k_2 = \frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta \{1 + 2(1 - l)^2\} > 0. \]

Pick \( \varepsilon_1 \) and \( \varepsilon_2 \) so small that (9) and (14) remain valid and

\[ \frac{1}{2} - \frac{\varepsilon_1 (1 - l)}{2 \xi l} - \varepsilon_2 \frac{g(0)}{4 \delta} C_p + \frac{(1 - l)}{\xi} \left( \frac{l}{2} + \frac{1}{2 \delta} \right) > 0 \]

Therefore (13) becomes

\[ F'(t) \leq -\beta \mathcal{E}(t), \quad \forall t \geq t_0. \]
We then use (9) to arrive at

\[ F'(t) \leq -\beta \alpha_1 F(t), \quad \forall t \geq t_0. \quad (16) \]

A simple integration of (16) leads to

\[ F(t) \leq F(t_0) e^{\beta \alpha_1 t_0} e^{-\beta \alpha_1 t}, \quad \forall t \geq t_0. \quad (17) \]

Again by the (9), (17) yields

\[ \mathcal{E}(t) \leq \alpha_2 F(t_0) e^{\beta \alpha_1 t_0} e^{-\beta \alpha_1 t}, \quad \forall t \geq t_0. \quad (18) \]

This completes the proof.

**Remark** By repeating the same procedure, the same result holds for

\[
\begin{align*}
 u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau \\
 + a(x)u_t + b|u|^{\gamma}u = 0, \quad b \neq 0 \text{ in } \Omega \times (0, \infty)
\end{align*}
\]