6. The line integral \( \int_{(1,0)}^{(3,4)} \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} \) is independent of path. Indeed it is of the form \( \int_{(1,0)}^{(3,4)} P\,dx + Q\,dy \), with \( P(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \) and \( Q(x, y) = \frac{y}{\sqrt{x^2 + y^2}} \) which are continuous and have partial derivatives continuous on any domain not containing the origin. Moreover we have \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \):

\[
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left[ x(x^2 + y^2)^{-1/2} \right] = x(-1/2)2y(x^2 + y^2)^{-3/2} = -xy(x^2 + y^2)^{-3/2},
\]

\[
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \left[ y(x^2 + y^2)^{-1/2} \right] = y(-1/2)2x(x^2 + y^2)^{-3/2} = -xy(x^2 + y^2)^{-3/2}.
\]

a) Since the integral is independent of path, there exists a function \( \phi \) such that \( d\phi = P\,dx + Q\,dy \) i.e.

\[
\frac{\partial \phi}{\partial x} = P(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \tag{1}
\]

\[
\frac{\partial \phi}{\partial y} = Q(x, y) = \frac{y}{\sqrt{x^2 + y^2}}. \tag{2}
\]

Integrating (1), we get

\[
\phi(x, y) = \int \frac{x\,dx}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2} + g(y). \tag{3}
\]

Using (2) and (3), we get

\[
\frac{y}{\sqrt{x^2 + y^2}} + g'(y) = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C. \tag{4}
\]
Combining (3) and (4), we obtain

\[ \phi(x, y) = \sqrt{x^2 + y^2} + C. \]

Hence

\[ \int_{(1,0)}^{(3,4)} \frac{xdy + ydx}{\sqrt{x^2 + y^2}} = \phi(3, 4) - \phi(1, 0) = \sqrt{3^2 + 4^2} - \sqrt{1^2 + 0^2} = \sqrt{25} - 1 = 4. \]

**b)** We consider the piecewise smooth curve \( C = C_1 \cup C_2 \) joining the points \((1, 0)\) and \((3, 4)\), where \( C_1 \) is the horizontal line segment joining the points \((1, 0)\) and \((3, 0)\), and where \( C_2 \) is the vertical line segment joining the points \((3, 0)\) and \((3, 4)\). \( C_1 \) and \( C_2 \) have the parameterizations

\[
C_1 : \begin{cases} 
    x = t, \\
    y = 0, \
\end{cases} \quad t \in [1, 3] \quad \text{and} \quad C_2 : \begin{cases} 
    x = 3, \\
    y = t, \
\end{cases} \quad t \in [0, 4].
\]

Then we have

\[
\int_{(1,0)}^{(3,4)} \frac{xdy + ydx}{\sqrt{x^2 + y^2}} = \int_C \frac{xdy + ydx}{\sqrt{x^2 + y^2}} = \int_{C_1} \frac{xdy + ydx}{\sqrt{x^2 + y^2}} + \int_{C_2} \frac{xdy + ydx}{\sqrt{x^2 + y^2}}. \tag{5}
\]

\[
\int_{C_1} \frac{xdy + ydx}{\sqrt{x^2 + y^2}} = \int_{C_1} \frac{x}{\sqrt{x^2 + y^2}} dx + \int_{C_1} \frac{y}{\sqrt{x^2 + y^2}} dy
= \int_1^3 \frac{t}{\sqrt{t^2 + 0^2}} dt + \int_1^3 0 dt
= \int_1^3 dt = [t]_1^3 = 2. \tag{6}
\]

\[
\int_{C_2} \frac{xdy + ydx}{\sqrt{x^2 + y^2}} = \int_{C_2} \frac{x}{\sqrt{x^2 + y^2}} dx + \int_{C_2} \frac{y}{\sqrt{x^2 + y^2}} dy
= \int_0^4 \frac{3}{\sqrt{9 + t^2}} (0) dt + \int_0^4 \frac{t}{\sqrt{9 + t^2}} dt = \int_0^4 \frac{t}{\sqrt{9 + t^2}} dt
= [\sqrt{9 + t^2}]_0^4 = [\sqrt{25} - \sqrt{9}] = 5 - 3 = 2. \tag{7}
\]

Using (5), (6) and (7), we get
\[
\int_{(1,0)}^{(3,4)} \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = 2 + 2 = 4.
\]

15. Let \( \mathbf{F}(x, y) = (x^3 + y)i + (x + y^3)j \) be a vector field. Since the functions \( P \) and \( Q \) are continuous and have partial derivatives continuous on any domain and moreover we have \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1 \), \( \mathbf{F} \) is a gradient field i.e. there exists a function \( \phi \) such that \( \nabla \phi(x, y) = \mathbf{F}(x, y) \) i.e.

\[
\frac{\partial \phi}{\partial x} = P(x, y) = x^3 + y \quad (1)
\]
\[
\frac{\partial \phi}{\partial y} = Q(x, y) = x + y^3. \quad (2)
\]

Integrating (1), we get

\[
\phi(x, y) = \int (x^3 + y)dx = \frac{1}{4}x^4 + xy + g(y). \quad (3)
\]

Using (2) and (3), we get

\[
x + g'(y) = x + y^3 \Rightarrow g'(y) = y^3 \Rightarrow g(y) = \frac{1}{4}y^4 + C. \quad (4)
\]

Combining (3) and (4), we obtain

\[
\phi(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + xy + C.
\]

18. Let \( \mathbf{F}(x, y) = (2x + e^{-y})i + (4y - xe^{-y})j \). We would like to evaluate the work \( W \) done by the force \( \mathbf{F} \) along the part of the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) located above the \( x \)-axis.

We remark that the components \( P \) and \( Q \) of the vector field \( \mathbf{F} \) are continuous and have partial derivatives continuous on any domain and moreover we have...
\( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -e^{-y}. \) Hence \( \mathbf{F} \) is a gradient field i.e. there exists a function \( \phi \) such that \( \nabla \phi(x, y) = \mathbf{F}(x, y) \) i.e.

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= P(x, y) = 2x + e^{-y} \\
\frac{\partial \phi}{\partial y} &= Q(x, y) = 4y - xe^{-y}.
\end{align*}
\]

Integrating (1), we get

\[ \phi(x, y) = \int (2x + e^{-y})dx = x^2 + xe^{-y} + g(y). \] (3)

Using (2) and (3), we get

\[ -xe^{-y} + g'(y) = 4y - xe^{-y} \quad \Rightarrow \quad g'(y) = 4y \quad \Rightarrow \quad g(y) = 2y^2 + C. \] (4)

Combining (3) and (4), we obtain

\[ \phi(x, y) = x^2 + xe^{-y} + 2y^2 + C. \]

Hence we have

\[
W = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} ((2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j})
\]

\[ = \phi(-2, 0) - \phi(2, 0) = (4 - 2e^0 + 0) - (4 + 2e^0 + 0) = -4. \]

24. The line integral \( \int_{(0,0,0)}^{(2,3,1)} 2xzdx + 2yzdy + (x^2 + y^2)dz \) is independent of path. Indeed it is of the form \( \int_{(0,0,0)}^{(2,3,1)} Pdx + Qdy + Rdz, \) with \( P(x, y, z) = 2xz, Q(x, y, z) = 2yz \) and \( R(x, y, z) = x^2 + y^2 \) which are continuous and have partial derivatives continuous on any domain. Moreover we have

\[
\begin{align*}
\frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 2x \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 2y.
\end{align*}
\]
Since the integral is independent of path, there exists a function $\phi$ such that $d\phi = Pdx + Qdy + Rdz$ i.e.

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= P(x, y, z) = 2xz \\
\frac{\partial \phi}{\partial y} &= Q(x, y, z) = 2yz \\
\frac{\partial \phi}{\partial z} &= R(x, y, z) = x^2 + y^2.
\end{align*}
\]

Integrating (1), we get

\[
\phi(x, y, z) = \int 2xz\,dx = x^2z + g(y, z). \quad (4)
\]

Using (2) and (4), we get

\[
\frac{\partial g}{\partial y} = 2yz \Rightarrow g(y, z) = y^2z + h(z) \Rightarrow \phi(x, y, z) = x^2z + y^2z + h(z). \quad (5)
\]

Using (3) and (5), we get

\[
x^2 + y^2 + h'(z) = x^2 + y^2 \Rightarrow h'(z) = 0 \Rightarrow h(z) = C. \quad (6)
\]

Combining (5) and (6), we obtain

\[
\phi(x, y, z) = x^2z + y^2z + C.
\]

Hence

\[
\int_{(2,0,0)}^{(0,0,0)} 2xz\,dx + 2yz\,dy + (x^2+y^2)\,dz = \phi(0,0,0) - \phi(2,3,1) = 0 - (-2)^2 - 3^2 = -4 - 9 = -13.
\]

28. Let $\mathbf{F}(x, y, z) = 8xy^3z\mathbf{i} + 12x^2y^2z\mathbf{j} + 4x^2y^3\mathbf{k}$. We would like to evaluate the works $W_1$ and $W_2$ done by the force $\mathbf{F}$ acting along the helix $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + tk$ from $(2,0,0)$ to $(1,\sqrt{3},\pi/3)$ and from $(2,0,0)$ to $(0,2,\pi/2)$ respectively.
We remark that the components \( P, Q \) and \( R \) of the vector field \( \mathbf{F} \) are continuous and have partial derivatives continuous on any domain and satisfy

\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 24xy^2z, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 8xy^3 \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 12x^2y^2.
\]

We deduce that the integral is independent of path. Moreover there exists a function \( \phi \) such that \( \nabla \phi = \mathbf{F} \) or equivalently \( d\phi = P\,dx + Q\,dy + R\,dz \) i.e.

\[
\begin{align*}
    \frac{\partial \phi}{\partial x} &= P(x, y, z) = 8xy^3z \quad (1) \\
    \frac{\partial \phi}{\partial y} &= Q(x, y, z) = 12x^2y^2z \quad (2) \\
    \frac{\partial \phi}{\partial z} &= R(x, y, z) = 4x^2y^3. \quad (3)
\end{align*}
\]

Integrating (1), we get

\[
\phi(x, y, z) = \int 8xy^3z\,dx = 4x^2y^3z + g(y, z). \quad (4)
\]

Using (2) and (4), we get

\[
12x^2y^2z + \frac{\partial g}{\partial y} = 12x^2y^2z \quad \Rightarrow \quad \frac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad g(y, z) = h(z)
\]

\[
\Rightarrow \quad \phi(x, y, z) = 4x^2y^3z + h(z). \quad (5)
\]

Using (3) and (5), we get

\[
4x^2y^3 + h'(z) = 4x^2y^3 \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h(z) = C. \quad (6)
\]

Combining (5) and (6), we obtain

\[
\phi(x, y, z) = 4x^2y^3z + C.
\]

Hence

\[
W_1 = \int_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} \mathbf{F}.d\mathbf{r} = \phi(1, \sqrt{3}, \pi/3) - \phi(2, 0, 0) = 4(1)^2(\sqrt{3})^3(\pi/3) - 0 = 4\pi\sqrt{3}
\]

and

\[
W_2 = \int_{(2,0,0)}^{(0,2,\pi/2)} \mathbf{F}.d\mathbf{r} = \phi(0, 2, \pi/2) - \phi(2, 0, 0) = 0 - 0 = 0.
\]

\[ \square \]