Directional fields

Objectives

1. What are direction fields?
2. Why do we need to learn to plot direction fields?
3. Plotting direction fields using MATLAB
4. Example
5. Direction fields and solutions on same graph window
1. What are direction fields?

Consider the 1st order differential equation
\[
\frac{dy}{dx} = f(x, y) \quad \text{(*)}
\]

- Let \( y(x) \) be a solution of \( \frac{dy}{dx} = f(x, y) \) whose graph is a solution curve of ODE.

- If \((x_0, y_0)\) is a point on solution curve, then the **slope of the tangent line to the solution curve** at \((x_0, y_0)\) is given by

\[
f(x_0, y_0) \quad \{\text{since } \frac{dy}{dx}\bigg|_{(x_0, y_0)} = f(x_0, y_0)\}
\]

**Key point:** we can get information about slope, of the solution curve, at any point directly from Eq. (\(\ast\)) \{without solving it\}.

- Hence, **we can draw small line segments with slope** \( f(x_i, y_i) \) **at any desired point** \((x_i, y_i)\). \{Usually we take a reasonably good collection of points to draw these lines\}.

- **The set of all these line segments is called a direction field (or slope field);** because at each point \((x_i, y_i)\) it gives direction (or slope) of the tangent line, to the solution curve, at \((x_i, y_i)\).
2. Why do we need to learn to plot direction fields

- Many times it will be impossible or too difficult to solve ODE of the type given by Eq. (*).

- But by plotting direction fields, you can get a good (geometric) idea about the solution and its properties.

- It is extremely useful because
  - Plotting direction fields is quite easy (you don’t have to solve ODE, you use it as it is).
  - It gives a pretty good idea of how the solution should look (because a solution curve is tangent to “small line segments” that are plotted)
  - You can use it as an aid to verify your approximate or numerical solutions.
3. Plotting direction fields in MATLAB

a) **Aim**
   To plot direction fields of
   \[
   \frac{dy}{dx} = f(x, y) \quad (*)
   \]

b) **What do we need to do**
   - Choose points where we want to draw slope fields
   - Find the slopes at these points directly from Eq. (*).
   - Draw the slope fields

c) **We need two main commands of MATLAB**
   - **meshgrid**
     \[
     \text{>> } [x,y]=\text{meshgrid}(a:k:b,c:j:d);
     \]
     creates a set of points (x,y) where
     i. ‘x’ lies between ‘a’ & ‘b’, incremented by ‘k’.
     ii. ‘y’ lies between ‘c’ & ‘d’, incremented by ‘j’.

     e.g. \[
     \text{>> } [x,y]=\text{meshgrid}(1:0.5:2,0:1:2);
     \]
     creates the set of nine points
     (1,0), (1,1), (1,2),
     (1.5,0), (1.5,1), (1.5,2),
     (2,0), (2,1), (2,2).
     See explanation in class

   - **quiver**
     \[
     \text{>> quiver(a,b,x,y)}
     \]
     begins at the point (a,b) and plots the vector v=(x,y).
4. Example

a) **Aim**

To plot direction fields of \( \frac{dy}{dx} = e^{-x} - 2y \)
on the rectangle \(-2 \leq x \leq 3, \ -1 \leq y \leq 2\).

b) **Can be done using following MATLAB commands**

\[
\begin{align*}
[x,y] &= \text{meshgrid}(-2:0.2:3,-1:0.2:2); \\
dy &= \text{exp}(-x) - 2*y; \\
dx &= \text{ones(size(dy))}; \\
\text{quiver}(x,y,dx,dy);
\end{align*}
\]

These commands generate the following.

Note: This is really an ugly picture.

One thing we observe that arrows have different lengths. Since we are only concerned with
direction, it is a good idea to make all vectors of unit length.

For this purpose, instead of before using “quiver” we continue with following commands
\begin{verbatim}
>> dyu=dy./sqrt(dx.^2+dy.^2);
>> dxu=dx./sqrt(dx.^2+dy.^2);
>> quiver(x,y,dxu,dyu);
>> axis equal tight
\end{verbatim}

With these commands we get the following.

We can do better using the command \texttt{>>quiver(x,y,dxu,dyu,0.5); (why)} which gives

See file dir1.m for all commands
### 5. Direction fields & solutions on same graph window

- Once you have plotted the direction field you can plot the solution curves (if you know the general solution) or your numerical solution on the same graph window using “hold on” and “hold off” as explained earlier.
- The general solution of the ODE in our example is \( y = e^{-x} + Ce^{-2x} \). The following figure shows direction fields with some of these solutions.

See file dir2.m (downloadable from course page) for all commands

End of lecture