1 Section 7.3 Vectors

Scalar quantities have a magnitude (numerical and unit description) only like area, distance and speed. Vector quantities have a magnitude and a direction like force and velocity.

A vector is a directed line segment. The length of the line segment is the magnitude of the vector, and the direction of the vector is measured by an angle.

The point A is called the initial point (or tail) of the vector and the point B is called the terminal point (or head) of the vector. The vector is denoted by \( \overrightarrow{AB} \) and the magnitude is denoted by \( ||AB|| \).

Equivalent vectors have the same magnitude and the same direction.

Multiplying a vector by a positive real number (other than 1) changes the magnitude of the vector but not its direction. Multiplying a vector by a negative real number \( a \) reverses the direction of the vector and multiplies the magnitude of the vector by \( |a| \).

Vectors can be added graphically by using the parallelogram method or the triangle method.

In the triangle method, the tail of the vector \( U \) is placed at the head of the other vector \( V \). In the parallelogram method, the tails of the two vectors \( U \) and \( V \) are placed together.

\[ V - U = V + (-U) \]

Vectors in a coordinate Plane

A vector can be moved in the plane as long as the magnitude and the direction are not changed. If \( P_1(x_1, y_1) \) is the initial point of a vector and \( P_2(x_2, y_2) \) is the terminal point, then an equivalent vector \( OP \) has its initial point at the origin and its terminal point at \( P(a, b) \) where \( a = x_2 - x_1 \) and \( b = y_2 - y_1 \). The vector can be denoted by \( v = \langle a, b \rangle \); \( a \) and \( b \) are called the components of the vector.

Example 1 Find the components of a vector \( CD \) whose tail is the point \( C(2, 5) \) and whose head is the point \( D(3, -1) \). Determine a vector \( w \) that is equivalent to \( CD \) and has its initial point at the origin.

The magnitude of the vector \( v = \langle a, b \rangle \) is \( ||V|| = \sqrt{a^2 + b^2} \). The direction angle is the angle between the vector and the positive x-axis and we can find it by \( \tan \alpha = \frac{b}{a} \).

Fundamental Vector Operations

If \( v = \langle a, b \rangle \) and \( w = \langle c, d \rangle \) are two vectors and \( k \) is a real number, then

1. \( ||v|| = \sqrt{a^2 + b^2} \)
2. \( v + w = \langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle \)
3. \( kv = k\langle a, b \rangle = \langle ka, kb \rangle \)

Example 2 Given \( v = (12, -5) \) and \( w = (2, 7) \), find 1) \( ||v|| \) 2) \( v + w \) 3) \( -5v \) 4) \( ||3v - 4w|| \)
A unit vector is a vector whose magnitude is 1.

Example 3 Which of the following is a unit vector? 1) \( v = (1, 1) \) 2) \( v = (0, 1) \) 3) \( v = \left( \frac{3}{2}, \frac{4}{3} \right) \)

A unit vector in the direction of a vector \( v \) is \( \frac{v}{||v||} \).

Example 4 Find a unit vector \( u \) in the direction of \( v = (2, -9) \).

Definition of Unit Vectors \( i \) and \( j \)
\( i = (1, 0) \) \( j = (0, 1) \)
If \( v \) is a vector and \( v = (a_1, a_2) \), then \( v = a_1 i + a_2 j \).

Example 5 If \( v = (3, 4) \), represent \( v \) in terms of the unit vectors \( i \) and \( j \).

Example 6 Given \( v = 4i + 3j \) and \( w = 6i - 3j \), find \( 4v + 5w \).

Horizontal and Vertical Components of a Vector
Let \( v = (a_1, a_2) \), where \( v \neq 0 \), the zero vector. Then \( a_1 = ||v|| \cos \theta \) and \( a_2 = ||v|| \sin \theta \) where \( \theta \) is the angle between the positive x-axis and \( v \).

The Horizontal component of \( v \) is \( ||v|| \cos \theta \). The Vertical component of \( v \) is \( ||v|| \sin \theta \).

Example 7 Is \( u = \cos \theta i + \sin \theta j \) a unit vector?

Example 8 Find the horizontal and vertical components of a vector \( v \) of magnitude 10 meters with direction angle 225°. Write the vector in the form \( v = a_1 i + a_2 j \).

Definition of Dot Product
Given \( v = (a, b) \) and \( w = (c, d) \), the dot product of \( v \) and \( w \) is given by \( v \cdot w = ac + bd \).

Example 9 Find the dot product of \( v = (6, -2) \) and \( w = (-2, 4) \).

In the following properties, \( u, v, \) and \( w \) are vectors and \( a \) is a scalar.
1) \( v \cdot w = w \cdot v \) 2) \( u \cdot (v + w) = u \cdot v + u \cdot w \) 3) \( a \cdot (u \cdot v) = (a \cdot u) \cdot v = u \cdot (a v) \)
4) \( v \cdot v = ||v||^2 \) 5) \( 0 \cdot v = 0 \) 6) \( i \cdot j = j \cdot i = 1 \) 7) \( i \cdot j = j \cdot i = 0 \)

Magnitude of a Vector in terms of the Dot Product
If \( v = (a, b) \), then \( ||v|| = \sqrt{v \cdot v} \)

Alternative Formula for the Dot Product
If \( v \) and \( w \) are two nonzero vectors and \( \alpha \) is the smallest non-negative angle between \( v \) and \( w \), then \( v \cdot w = ||v|| ||w|| \cos \alpha \).

Angle between Two Vectors
If \( v \) and \( w \) are two nonzero vectors and \( \alpha \) is the smallest non-negative angle between \( v \) and \( w \), then \( \cos \alpha = \frac{v \cdot w}{||v|| ||w||} \) and \( \alpha = \cos^{-1} \left( \frac{v \cdot w}{||v|| ||w||} \right) \).

Example 10 Find the measure of the smallest positive angle between the vectors \( v = 3i + 2j \) and \( w = -2i - j \).
Parallel and Perpendicular Vectors
Two vectors are parallel when the angle $\alpha$ between the vectors is $0^\circ$ or $180^\circ$.
Two nonzero vectors $v$ and $w$ are orthogonal (perpendicular) if and only if $v \cdot w = 0$.

Scalar Projection
If $v$ and $w$ are two nonzero vectors and $\alpha$ is the smallest non-negative angle between $v$ and $w$, then the scalar projection of $v$ on $w$, $\text{proj}_w v$, is given by $\text{proj}_w v = ||v|| \cos \alpha$.

Example 11  Given $v = 3i - 4j$ and $w = i + 3j$, find $\text{proj}_w v$.

Example 12  Find a vector that has the initial point $(3, -1)$ and is equivalent to $v = 2i - 3j$.

Example 13  Let $w = 4i + j$. Find a vector perpendicular to $w$.