Section 5.3 Trigonometric Functions of Any Angle

Let $P(x, y)$ be any point, except the origin, on the terminal side of an angle $\theta$ in standard position. Let $r = d(O, P)$, the distance from the origin to $P$. The six trigonometric functions of $\theta$ are

$$
\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}, \quad x \neq 0 \quad \csc \theta = \frac{r}{y}, \quad y \neq 0 \quad \sec \theta = \frac{r}{x}, \quad x \neq 0 \quad \cot \theta = \frac{x}{y}, \quad y \neq 0
$$

where $r = \sqrt{x^2 + y^2}$.

Note that the value of any trigonometric function is independent of the point chosen on the terminal side of the angle.

Example 1 Find the exact value of the six trigonometric functions of an angle in standard position whose terminal side contains the point $(4, -1)$.

Values of Trigonometric Functions for Quadrantal Angles

The terminal side of $0^\circ$ coincides with the positive x-axis. Let $P(x, 0)$, $x > 0$, be any point on the x-axis. Then $y = 0$, and $r = x$. The value of the six trigonometric functions of $0^\circ$ are $\sin 0^\circ = \frac{0}{r} = 0$, $\cos 0^\circ = 1$, $\tan 0^\circ = 0$, $\csc 0^\circ$ is undefined, $\sec 0^\circ = 1$, and $\cot 0^\circ$ is undefined.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
<th>$\csc \theta$</th>
<th>$\sec \theta$</th>
<th>$\cot \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
<td>1</td>
<td>undefined</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
<td>1</td>
<td>undefined</td>
<td>0</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>undefined</td>
<td>-1</td>
<td>undefined</td>
</tr>
<tr>
<td>$270^\circ$</td>
<td>-1</td>
<td>0</td>
<td>undefined</td>
<td>-1</td>
<td>undefined</td>
<td>0</td>
</tr>
</tbody>
</table>

The sign of a trigonometric function depends on the quadrant in which the terminal side of the angle lies. For example, if $\theta$ is an angle whose terminal side lies in Quadrant II and $P(x, y)$ is on the terminal side of $\theta$, then both $x$ and $y$ are negative.

Signs of the Trigonometric Functions

<table>
<thead>
<tr>
<th>sign of</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$ and $\csc \theta$</td>
<td>positive</td>
<td>positive</td>
<td>negative</td>
<td>negative</td>
</tr>
<tr>
<td>$\cos \theta$ and $\sec \theta$</td>
<td>positive</td>
<td>negative</td>
<td>negative</td>
<td>positive</td>
</tr>
<tr>
<td>$\tan \theta$ and $\cot \theta$</td>
<td>positive</td>
<td>negative</td>
<td>positive</td>
<td>negative</td>
</tr>
</tbody>
</table>

Example 2 Determine each of the following: 1) $\sin 0^\circ$ 2) $\cos 90^\circ$ 3) $\sin 180^\circ$ 4) $\cos 270^\circ$.

Example 3 True or False: 1) If $\sin \theta$ is positive, then $\theta$ must be in Quadrant I.

2) If $\sin \theta$ is negative, then $\theta$ must be in either Quadrant III or Quadrant IV.

3) If $\sin \theta$ is negative and $\cos \theta$ is positive, then $\theta$ must be in Quadrant IV.
Given sec $\theta = 3$ and sin $\theta < 0$, find tan $\theta$ and csc $\theta$.

**Reference Angle**

Given $\angle \theta$ in standard position, its reference angle $\theta'$ is the smallest positive angle formed by the terminal side of $\angle \theta$ and the x-axis.

**Example 4** For each of the following angles $\theta$ determine the measures of its reference angle $\theta'$. 1) $\theta = 135^\circ$  2) $\theta = 330^\circ$  3) $\theta = \frac{10\pi}{3}$  4) $\theta = -6$  5) $\theta = 30$

**Reference Angle Theorem**

To evaluate $\sin \theta$, determine $\sin \theta'$. Then use either $\sin \theta'$ or its opposite as the answer, depending on which has the correct sign.

**Example 5** True or False: 1) $\sin 150^\circ = \sin 30^\circ$  2) $\cos 150^\circ = \cos 30^\circ$  3) $\tan 150^\circ = \tan 30^\circ$  4) $\sin(-45^\circ) = -\sin 45^\circ$  5) $\cos(-45^\circ) = \cos 45^\circ$

**Example 6** Determine the exact value of each function

1) $\sin 330^\circ$  2) $\cos 660^\circ$  3) $\tan 240^\circ$  4) $\csc(-510^\circ)$

**Example 7** Find the exact value of each expression 1) $\tan 225^\circ + \sin 240^\circ \cos 60^\circ$  2) $\sin^2 \left(\frac{5\pi}{4}\right) + \cos^2 \left(\frac{5\pi}{4}\right)$