1 Section 3.1 Polynomial Division and Synthetic Division

Polynomial Division
Dividing a polynomial by another polynomial is similar to the long division process used for diving positive integers.

Example 1 Divide \((x^2 + 9x + 12)\) by \((x + 4)\)

\[
\begin{array}{llll}
\text{Dividend} & x^2 + 9x + 12 & \text{Divisor} & x + 4 \\
\text{Quotient} & x + 5 & \text{Remainder} & -8 \\
\end{array}
\]

The dividend is equal to the product of the divisor and the quotient plus the remainder.

\(x^2 + 9x + 12 = (x + 4)(x + 5) - 8.\)

The Division Algorithm for Polynomials
If \(P(x)\) and \(D(x)\) are polynomials such that \(D(x) \neq 0\), then there exist unique polynomials \(Q(x)\) and \(R(x)\) such that \(P(x) = D(x)Q(x) + R(x)\), where either \(R(x) = 0\) or the degree of \(R(x)\) is less than the degree of \(D(x)\).

\[
P(x) = D(x)Q(x) + R(x).
\]

Example 2 Perform the indicated division
1) \[
\frac{x^4 + x^3 + 5x^2 + 4x + 4}{x^2 + 3x + 5}
\]
2) \[
\frac{x^4 - 4x^3 + 10x^2 + 12}{x - 3}
\]

Synthetic Division
We can divide a polynomial by a binomial of the form \(x - c\), by a method called a synthetic division.

If we apply the long division for \(\frac{3x^3 - 8x^2 + 7x + 2}{x - 2}\), and try to do the following steps, we will get the synthetic division.
1) Omit the variables.
2) Omit the repeated coefficients.
3) Change the sign of the divisor.

Example 3 Use synthetic division to perform the indicated operation \(\frac{x^4 - 4x^3 + 10x^2 + 12}{x - 3}\).

The Remainder Theorem
If a polynomial \(P(x)\) is divided by \(x - c\), then the remainder is \(P(c)\).

Example 4 Use the remainder theorem to evaluate \(P(x) = 3x^3 + 5x^2 - 11x + 3\) when \(x = -3\) and \(x = 2\).

Zero of a polynomial
If \(P(x)\) is a polynomial and \(a\) is a number for which \(P(a) = 0\), then \(a\) is a zero of \(P(x)\).

The Factor Theorem
A polynomial \(P(x)\) has a factor \((x - c)\) if and only if \(P(c) = 0\). That is \((x - c)\) is a factor of \(P(x)\) if and only if \(c\) is a zero of \(P(x)\).

Example 5 Determine whether \((x + 2)\) is a factor of \(P(x) = x^3 + 3x^2 + 4x + 4\).
Example 6 Determine whether \((x - 2)\) is a factor of \(P(x) = x^3 + 3x^2 + 4x + 4\).

\[
P(x) = x^3 + 3x^2 + 4x + 4 = (x + 2)(x^2 + x + 2)
\]

Reduced Polynomials

\(Q(x) = x^2 + x + 2\) is called a reduced polynomial or a depressed polynomial because it is one degree less than the degree of \(P(x)\).

Example 7 Verify that \((x + 4)\) is a factor of \(P(x) = 3x^4 + 11x^3 - 6x^2 - 6x + 8\), and write \(P(x)\) as the product of \((x + 4)\) and the reduced polynomial \(Q(x)\).

Example 8 Use the factor theorem to prove that for any positive odd integer \(n\), \(x^n - 1\) has \(x - 1\) as a factor.

Example 9 Find the remainder of \(5x^{48} + 6x^{10} - 5x + 7\) divided by \(x - 1\).

Example 10 Use synthetic division to show that \((x - i)\) is a factor of \(x^3 + 3x^2 + x - 3\).

Example 11 Find the values of \(k\) so that when \(x^2 - 3x - 8\) is divided by \(x + k\), the remainder is equal to \(-4\).

Example 12 If \(x^4 + 2x^3 - 2x - 2 = (x - 1)(x + 1)g(x) - 1\), then find \(g(x)\).

Example 13 If \(P(x) = x^{105} - x^{10} - 2x + 1\) is divided by \(x - i\), then find the remainder.

\[
\begin{array}{cccc}
i & 1 & i & m & 2 \\
i & i & n & w & 2 + i \\
\end{array}
\]

where \(i = \sqrt{-1}\) is the synthetic division of some Polynomial \(P(x)\) by \(x - i\), then find the quotient.